REPRESENTATION THEORY OF SUPERCONFORMAL ALGEBRAS AND THE KAC-ROAN-WAKIMOTO CONJECTURE

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Abstract. We study the representation theory of the superconformal algebra $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ associated with a minimal gradation of $\mathfrak{g}$. Here, $\mathfrak{g}$ is a simple finite-dimensional Lie superalgebra with a non-degenerate, even supersymmetric invariant bilinear form. Thus, $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ can be one of the well-known superconformal algebras including the Virasoro algebra, the Bershadsky-Polyakov algebra, the Neveu-Schwarz algebra, the Bershadsky-Knizhnik algebras, the $N = 2$ superconformal algebra, the $N = 4$ superconformal algebra, the $N = 3$ superconformal algebra and the big $N = 4$ superconformal algebra. We prove the conjecture of V. G. Kac, S.-S. Roan and M. Wakimoto for $\mathcal{W}_k(\mathfrak{g}, f_\theta)$. In fact, we show that any irreducible highest weight character of $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ at any level $k \in \mathbb{C}$ is determined by the corresponding irreducible highest weight character of the Kac-Moody affinization of $\mathfrak{g}$.

1. Introduction

Suppose that the following are is given: (i) a simple finite-dimensional Lie superalgebra $\mathfrak{g}$ with non-degenerate, even supersymmetric invariant bilinear form, (ii) a nilpotent element f in the even part of $\mathfrak{g}$ and (iii) a level $k \in \mathbb{C}$ of the Kac-Moody affinization $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$. Then, the corresponding $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ can be constructed using the method of quantum BRST reduction. This method was first introduced by B. L. Feĭgin and E. V. Frenkel [10, 11] in the case that $\mathfrak{g}$ is a Lie algebra and f is its principal nilpotent element, and it was recently extended to the general case by V. G. Kac, S.-S. Roan and M. Wakimoto [17].

In this paper we study the representation theory of those $\mathcal{W}$-algebras for which the nilpotent element f is equal to the root vector f_θ corresponding to the lowest root $-\theta$ of $\mathfrak{g}$. V. G. Kac, S.-S. Roan and M. Wakimoto [17] showed that these $\mathcal{W}$-algebras $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ are quite different from the standard $\mathcal{W}$-algebras [29, 8, 24, 7, 25, 10, 11] associated with principal nilpotent elements and noteworthy because they include almost all the superconformal algebras so far constructed to this time by physicists such as the Virasoro algebra, the Bershadsky-Polyakov algebra [5], the Neveu-Schwarz algebra, the Bershadsky-Knizhnik algebras [4], the $N = 2$ superconformal algebra, the $N = 4$ superconformal algebra, the $N = 3$ superconformal algebra and the big $N = 4$ superconformal algebra.

Let $\mathcal{O}_k$ be the Bernstein-Gel'fand-Gel'fand category of $\widehat{\mathfrak{g}}$ at level $k \in \mathbb{C}$. Let $M(\lambda)$ be the Verma module of $\widehat{\mathfrak{g}}$ with highest weight λ and $L(\lambda)$ be the unique irreducible quotient of $M(\lambda)$. The method of quantum BRST reduction gives a family of functors $V \rightsquigarrow H^i(V)$, depending on $i \in \mathbb{Z}$, from $\mathcal{O}_k$ to the category of $\mathcal{W}_k(\mathfrak{g}, f)$-modules. Here, $H^\bullet(V)$ is the BRST cohomology of the corresponding

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quantum reduction. In the case that $\mathfrak{g}$ is a Lie algebra and f is its principal nilpotent element, E. V. Frenkel, V. G. Kac and M. Wakimoto [13] used this functor in their construction of the "minimal" series presentations of $\mathcal{W}_k(\mathfrak{g}, f)$, and they conjectured that $H^\bullet(L(\lambda))$ is irreducible (or zero) for an admissible weight λ. This conjecture was extended by V. G. Kac, S.-S. Roan and M. Wakimoto [17] to the general case, in which they conjectured that, for an admissible weight λ, the irreduciblity of $H^\bullet(L(\lambda))$ holds for a general pair $(\mathfrak{g}, f)$ consisting of a Lie superalgebra $\mathfrak{g}$ and a nilpotent element f (see Conjecture 3.1B of Ref. [17]).

As a continuation of the present author's previous work [1, 2], in which the conjecture of E. V. Frenkel, V. G. Kac and M. Wakimoto was proved (completely for the "−" case and partially for the "+" case), we prove the conjecture of V. G. Kac, S.-S. Roan and M. Wakimoto for $\mathcal{W}_k(\mathfrak{g}, f_\theta)$; Actually we prove even stronger results, showing that the representation theory of $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is controlled by $\widehat{\mathfrak{g}}$ in the following sense:

Main Theorem. *For arbitrary level $k \in \mathbb{C}$, we have the following:*

(1) (Theorem 6.7.1) We have $H^i(V) = \{0\}$ with $i \neq 0$ for any $V \in Obj\mathcal{O}_k$.
(2) (Theorem 6.7.4) Let $L(\lambda) \in Obj\mathcal{O}_k$ be the irreducible $\widehat{\mathfrak{g}}$-module with highest weight λ. Then, $\langle \lambda, \alpha_0^\vee \rangle \in \{0, 1, 2, \dots\}$ implies $H^0(L(\lambda)) = \{0\}$. Otherwise, $H^0(L(\lambda))$ is isomorphic to the irreducible $\mathcal{W}_k(\mathfrak{g}, f_\theta)$-module with the corresponding highest weight.

Main Theorem (1) implies, in particular, that the correspondence $V \rightsquigarrow H^0(V)$ defines an exact functor from $\mathcal{O}_k$ to the category of $\mathcal{W}_k(\mathfrak{g}, f_\theta)$-modules, defining a map between characters. On the other hand, every irreducible highest weight representation of $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is isomorphic to $H^0(L(\lambda))$ for some λ (see Section 6). It is also known that $H^0(M(\lambda))$ is a Verma module over $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ [19]. Hence, from the above results, it follows that the character of any irreducible highest weight representation of $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is determined by the character of the corresponding irreducible $\widehat{\mathfrak{g}}$-module $L(\lambda)$. We remark that Main Theorem (2) is consistent with the computation of V. G. Kac, S.-S. Roan and M. Wakimoto [17] of the Euler-Poincaré character of $H^\bullet(L(\lambda))$.

This paper is organized as follows. In Section 2 we collect the necessary information regarding the affine Lie superalgebra $\widehat{\mathfrak{g}}$. In this setting, a slight modification is need for the $A(1,1)$ case, which is summarized in Appendix. In Section 3 we recall the definition of the BRST complex constructed by V. G. Kac, S.-S. Roan, and M. Wakimoto [17]. As explained in Ref. [17], their main idea in generalizing the construction of B. L. Feĭgin and E. V. Frenkel [10, 11] was to add the "neutral free superfermions" whose definition is given at the beginning of that section. Although the $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ can be defined for an arbitrary nilpotent element f, the assumption $f = f_\theta$ simplifies the theory in many ways. This is also the case that the above-mentioned well-known superconformal algebras appear, as explained in Ref. [17]. In Section 4 we derive some basic but important facts concerning the BRST cohomology under the assumption $f = f_\theta$. In Section 5 we recall the definition of the $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ and collect necessary information about its structure. In Section 6 we present the parameterization of irreducible highest weight representations of $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ and state our main results (Theorems 6.7.1 and 6.7.4). The most important part of the proof is the computation of the BRST cohomology $H^\bullet(M(\lambda)^*)$

associated with the dual $M(\lambda)^*$ of the Verma module $M(\lambda)$. This is carried out in Section 7 by introducing a particular spectral sequence. The argument used here is a modified version of that given in Ref. [1], where we proved the vanishing of the cohomology associated to the original quantum reduction formulated by B. L. Feĭgin and E. V. Frenkel [10, 11, 13].

The method used in this paper can also be applied to general $\mathcal{W}$-algebras, with some modifications. Our results for that case will appear in forthcoming papers.

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2. Preliminaries

2.1. Let $\mathfrak{g}$ be a complex, simple finite-dimensional Lie superalgebra with a non-degenerate, even supersymmetric invariant bilinear form $(.|.)$. Let (e, x, f) be a $\mathfrak{sl}_2$-triple in the even part of $\mathfrak{g}$ normalized as

$$[e,f]=x,\ \ [x,e]=e,[x,f]=-f. \tag{1}$$

Further, let

$$\mathfrak{g}=\bigoplus_{j\in\frac{1}{2}\mathbb{Z}}\mathfrak{g}_j,\quad \mathfrak{g}_j=\{u\in\mathfrak{g}\mid [x,u]=ja\} \tag{2}$$

be the eigenspace decomposition of $\mathfrak{g}$ with respect to $\operatorname{ad} x$.

2.2. Define $\mathfrak{g}^f$ as the centralizer of f in $\mathfrak{g}$, so that $\mathfrak{g}^f=\{u\in\mathfrak{g}\mid [f,u]=0\}$. Then, we have $\mathfrak{g}^f=\sum_{j\le 0}\mathfrak{g}^f_j$, where $\mathfrak{g}^f_j=\mathfrak{g}^f\cap\mathfrak{g}_j$. Similarly, we define $\mathfrak{g}^e \underset{\text{def}}{=} \{u\in\mathfrak{g}\mid [e,u]=0\}=\sum_{j\ge 0}\mathfrak{g}^e_j$.

2.3. Define the following:

$$\mathfrak{g}_{\ge 1}\underset{\text{def}}{=}\bigoplus_{j\ge 1}\mathfrak{g}_j,\quad \mathfrak{g}_{>0}\underset{\text{def}}{=}\bigoplus_{j>0}\mathfrak{g}_j.$$

Then, $\mathfrak{g}_{>0}=\mathfrak{g}_{\ge 1}\oplus\mathfrak{g}_{\frac{1}{2}}$, and $\mathfrak{g}_{\ge 1}$ and $\mathfrak{g}_{>0}$ are both nilpotent subalgebras of $\mathfrak{g}$. Similarly define $\mathfrak{g}_{\ge 0}$, $\mathfrak{g}_{\le 0}$, $\mathfrak{g}_{<0}$ and $\mathfrak{g}_{\le -1}$.

2.4. Define the character $\bar{\chi}$ of $\mathfrak{g}_{\ge 1}$ by

$$\bar{\chi}(u)\underset{\text{def}}{=}(f|u),\quad \text{where } u\in\mathfrak{g}_{\ge 1}. \tag{3}$$

Then, $\bar{\chi}$ defines a skew-supersymmetric even bilinear form $\langle .|.\rangle_{\text{ne}}$ on $\mathfrak{g}_{\frac{1}{2}}$ through the formula

$$\langle u|v\rangle_{\text{ne}}=\bar{\chi}([u,v]). \tag{4}$$

Note that $\langle .|.\rangle_{\text{ne}}$ is non-degenerate, as guaranteed by the $\mathfrak{sl}_2$-representation theory. Also, we have the property

$$\langle u|[a,v]\rangle_{\text{ne}}=\langle [u,a]|v\rangle_{\text{ne}}\quad \text{for } a\in\mathfrak{g}^f_0, u,v\in\mathfrak{g}_{\frac{1}{2}}. \tag{5}$$

2.5. Let $\mathfrak{h} \subset \mathfrak{g}_0$ be a Cartan subalgebra of $\mathfrak{g}$ containing x, and let $\Delta \subset \mathfrak{h}^*$ be the set of roots. It is known that the root space $\mathfrak{g}^{\alpha}$, with $\alpha \in \Delta$, is one-dimensional except for $A(1,1)$ (see [16]). For this reason, in the case of $A(1,1)$, a slight modification is needed in the following argument, which is summarized in Appendix. Now, let $\Delta_j = \{\alpha \in \Delta \mid \langle \alpha, x \rangle = j\}$ with $j \in \frac{1}{2}\mathbb{Z}$. Then, $\Delta = \sqcup_{j \in \frac{1}{2}\mathbb{Z}} \Delta_j$, and Δ_0 is the set of roots of $\mathfrak{g}_0$. Let Δ_{0+} be a set of positive roots of Δ_0. Then, $\Delta_+ = \Delta_{0+} \sqcup \Delta_{>0}$ is a set of positive roots of $\mathfrak{g}$, where $\Delta_{>0} = \sqcup_{j>0} \Delta_j$. This gives the triangular decompositions

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \quad \mathfrak{g}_0 = \mathfrak{n}_{0,-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{0,+}. \tag{6}$$

Here, $\mathfrak{n}_+ \underset{\text{def}}{=} \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$, $\mathfrak{n}_{0,+} \underset{\text{def}}{=} \sum_{\alpha \in \Delta_{0,+}} \mathfrak{g}_\alpha$ and analogously for $\mathfrak{n}_-$ and $\mathfrak{n}_{0,-}$.

2.6. Let $u \mapsto u^t$ be an anti-automorphism of $\mathfrak{g}$ such that $e^t = f$, $f^t = e$, $x^t = x$, $\mathfrak{g}_\alpha^t = \mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta$ and $(u^t|v^t) = (v|u)$ for $u, v \in \mathfrak{g}$. We fix the root vectors $u_\alpha \in \mathfrak{g}_\alpha$, where $\alpha \in \Delta$, such that $(u_\alpha, u_{-\alpha}) = 1$ and $u_\alpha^t = u_{-\alpha}$ with $\alpha \in \Delta_+$.

2.7. Let $p(\alpha)$ be the parity of $\alpha \in \Delta$ and $p(v)$ be the parity of $v \in \mathfrak{g}$.

2.8. Let $\widehat{\mathfrak{g}}$ be the *Kac–Moody affinization* of $\mathfrak{g}$. This is the Lie superalgebra given by

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D} \tag{7}$$

with the commutation relations

$$[u(m), v(n)] = [u, v](m+n) + m\delta_{m+n,0}(u|v)K, \tag{8}$$

$$[\mathbf{D}, u(m)] = mu(m), \quad [K, \widehat{\mathfrak{g}}] = 0 \tag{9}$$

for $u, v \in \mathfrak{g}$, $m, n \in \mathbb{Z}$. Here, $u(m) \underset{\text{def}}{=} u \otimes t^m$ for $u \in \mathfrak{g}$ and $m \in \mathbb{Z}$.

The invariant bilinear form $(\cdot|\cdot)$ is extended from $\mathfrak{g}$ to $\widehat{\mathfrak{g}}$ by stipulating $(u(m)|v(n)) = (u|v)\delta_{m+n,0}$ with $u, v \in \mathfrak{g}$, $m, n \in \mathbb{Z}$, $(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}], \mathbb{C}K \oplus \mathbb{C}\mathbf{D}) = 0$, $(K, K) = (\mathbf{D}, \mathbf{D}) = 0$ and $(K, \mathbf{D}) = (\mathbf{D}, K) = 1$.

2.9. Define the subalgebras

$$L\mathfrak{g}_{\geq 1} \underset{\text{def}}{=} \mathfrak{g}_{\geq 1} \otimes \mathbb{C}[t, t^{-1}], \quad L\mathfrak{g}_{>0} \underset{\text{def}}{=} \mathfrak{g}_{>0} \otimes \mathbb{C}[t, t^{-1}] \subset \widehat{\mathfrak{g}}. \tag{10}$$

Similarly, define $L\mathfrak{g}_{\geq 0}$, $L\mathfrak{g}_{\leq 0}$, $L\mathfrak{g}_{<0}$ and $L\mathfrak{g}_{\leq -1}$.

2.10. Fix the triangular decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+$ in the standard way. Then, we have

$$\begin{aligned}
&\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D}, \\
&\widehat{\mathfrak{n}}_- = \mathfrak{n}_- \otimes \mathbb{C}[t^{-1}] \oplus \mathfrak{h} \otimes \mathbb{C}[t^{-1}]t^{-1} \oplus \mathfrak{n}_+ \otimes \mathbb{C}[t^{-1}]t^{-1}, \\
&\widehat{\mathfrak{n}}_+ = \mathfrak{n}_- \otimes \mathbb{C}[t]t \oplus \mathfrak{h} \otimes \mathbb{C}[t]t \oplus \mathfrak{n}_+ \otimes \mathbb{C}[t].
\end{aligned}$$

Let $\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ be the dual of $\widehat{\mathfrak{h}}$, where, Λ_0 and δ are dual elements of K and $\mathbf{D}$, respectively. Next, let $\widehat{\Delta}$ be the set of roots of $\widehat{\mathfrak{g}}$, $\widehat{\Delta}_+$ the set of positive roots, and $\widehat{\Delta}_-$ the set of negative roots. Then, we have $\widehat{\Delta}_- = -\widehat{\Delta}_+$. Further, let $\widehat{Q}$ be the root lattice and $\widehat{Q}_+ = \sum_{\alpha \in \widehat{\Delta}_+} \mathbb{Z}_{\geq 0}\alpha \subset \widehat{Q}$. We define a partial ordering $\mu \leq \lambda$ on $\widehat{\mathfrak{h}}^*$ by $\lambda - \mu \in \widehat{Q}_+$.

2.11. For an $\widehat{\mathfrak{h}}$-module V, let V^λ be the weight space of weight λ, that is, $V^\lambda \underset{\text{def}}{=} \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \widehat{\mathfrak{h}}\}$. If all the weight spaces V^λ are finite-dimensional, we define the graded dual V^* of V by

$$V^* \underset{\text{def}}{=} \bigoplus_{\lambda \in \mathfrak{h}^*} \mathrm{Hom}_{\mathbb{C}}(V^\lambda, \mathbb{C}) \subset \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}). \tag{11}$$

2.12. Throughout this paper, k represents a complex number. Let $\widehat{\mathfrak{h}}^*_k$ denote the set of weights of $\widehat{\mathfrak{g}}$ of level k:

$$\widehat{\mathfrak{h}}^*_k \underset{\text{def}}{=} \{\lambda \in \widehat{\mathfrak{h}}^* \mid \langle \lambda, K\rangle = k\}. \tag{12}$$

Also, let $\mathcal{O}_k$ be the full subcategory of the category of left $\widehat{\mathfrak{g}}$-modules consisting of objects V such that the following hold:

(1) $V = \bigoplus_{\lambda \in \widehat{\mathfrak{h}}^*_k} V^\lambda$ and $\dim_{\mathbb{C}} V^\lambda < \infty$ for all $\lambda \in \widehat{\mathfrak{h}}^*_k$;
(2) there exists a finite set $\{\mu_1, \ldots, \mu_r\} \subset \widehat{\mathfrak{h}}^*_k$ such that $\lambda \in \bigcup_i (\mu_i - \widehat{Q}_+)$ for any weight λ with $V^\lambda \neq \{0\}$.

Then, $\mathcal{O}_k$ is an abelian category. Let $M(\lambda) \in Obj\mathcal{O}_k$, with $\lambda \in \widehat{\mathfrak{h}}^*_k$, be the Verma module with highest weight λ. That is, $M(\lambda) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+)} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one-dimensional $\widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+$-module on which $\widehat{\mathfrak{n}}_+$ acts trivially and $h \in \widehat{\mathfrak{h}}$ acts as $\langle \lambda, h\rangle\, \mathrm{id}$. Let v_λ be the highest weight vector of $M(\lambda)$. Next, let $L(\lambda) \in Obj\mathcal{O}_k$ be the unique simple quotient of $M(\lambda)$.

2.13. The correspondence $V \rightsquigarrow V^*$ defines the duality functor in $\mathcal{O}_k$. Here, $\widehat{\mathfrak{g}}$ acts on V^* as $(af)(v) = f(a^t v)$, where $a \mapsto a^t$ is the antiautomorphism of $\widehat{\mathfrak{g}}$ defined by $u(m)^t = (u^t)(-m)$ (with $u \in \mathfrak{g}, m \in \mathbb{Z}$), $K^t = K$ and $\mathbf{D}^t = \mathbf{D}$. We have $L(\lambda)^* = L(\lambda)$ for $\lambda \in \widehat{\mathfrak{h}}^*_k$.

2.14. Let $\mathcal{O}_k^\triangle$ be the full subcategory of $\mathcal{O}_k$ consisting of objects V that admit a Verma flag, that is, a finite filtration $V = V_0 \supset V_1 \supset \cdots \supset V_r = \{0\}$ such that each successive subquotient V_i/V_{i+1} is isomorphic to some Verma module $M(\lambda_i)$ with $\lambda_i \in \widehat{\mathfrak{h}}^*$. The category $\mathcal{O}_k^\triangle$ is stable under the operation of taking a direct summand. Dually, let $\mathcal{O}_k^\nabla$ be the full subcategory of $\mathcal{O}_k$ consisting of objects V such that $V^* \in Obj\mathcal{O}_k^\triangle$.

2.15. For $\lambda \in \widehat{\mathfrak{h}}^*_k$, let $\mathcal{O}_k^{\leq\lambda}$ be the full subcategory of $\mathcal{O}_k$ consisting of objects V such that $V = \bigoplus_{\mu \leq \lambda} V^\mu$. Then, $\mathcal{O}_k^{\leq\lambda}$ is an abelian category and stable under the operation of taking (graded) dual. Also, every simple object $L(\mu) \in Obj\mathcal{O}_k^{\leq\lambda}$, with $\mu \leq \lambda$, admits a projective cover $P_{\leq\lambda}(\mu)$ in $\mathcal{O}_k^{\leq\lambda}$, and hence, every finitely generated object in $\mathcal{O}_k^{\leq\lambda}$ is an image of a projective object of the form $\bigoplus_{i=1}^r P_{\leq\lambda}(\mu_i)$ for some $\mu_i \in \widehat{\mathfrak{h}}^*$. Indeed, as in the Lie algebra case (see, e.g., Ref. [27]), $P_{\leq\lambda}(\mu)$ can be defined as an indecomposable direct summand of

$$U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{g}}_+)} \tau_{\leq\lambda} \left(U(\widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+) \otimes_{U(\widehat{\mathfrak{h}})} \mathbb{C}_\mu \right)$$

that has $L(\mu)$ as a quotient. Here, $\tau_{\leq\lambda}(V) = V / \bigoplus_{\substack{\nu \in \widehat{\mathfrak{h}}^* \\ \nu \not\leq \lambda}} V^\nu$, and $\mathbb{C}_\mu$ is a one-dimensional $\widehat{\mathfrak{h}}$-module on which $h \in \widehat{\mathfrak{h}}$ acts as $\mu(h)\, \mathrm{id}$. Note that $P_{\leq\lambda}(\mu) \in Obj\mathcal{O}_k^\triangle$.

Moreover, the Bernstein-Gel'fand-Gel'fand reciprocity holds:

$$[P_{\leq\lambda}(\mu) : M(\mu')] = [M(\mu') : L(\mu)] \quad (\text{with } \mu, \mu' \leq \lambda).$$

Here, $[P_{\leq\lambda}(\mu) : M(\mu')]$ is the multiplicity of $M(\mu')$ in the Verma flag of $P_{\leq\lambda}(\mu)$, and $[M(\mu') : L(\mu)]$ is the multiplicity of $L(\mu)$ in the local composition factor (see Ref. [15]) of $M(\mu')$. Dually, $I_{\leq\lambda}(\mu) = P_{\leq\lambda}(\mu)^*$ is the injective envelope of $L(\mu)$ in $\mathcal{O}_k^{\leq\lambda}$. In particular, $V \in Obj\mathcal{O}_k^{\leq\lambda}$ is a submodule of an injective object of the form $\bigoplus_{i=1}^r I_{\leq\lambda}(\mu_i)$ for some $\mu_i \in \widehat{\mathfrak{h}}^*$ if its dual V^* is finitely generated.

3. The Kac-Roan-Wakimoto construction I: the BRST complex

3.1. Define a character χ of $L\mathfrak{g}_{\geq 1}$ by

$$\chi(u(m)) \underset{\text{def}}{=} (f(1)|u(m)) = \bar{\chi}(u)\delta_{m,-1} \quad \text{for } u \in \mathfrak{g}_{\geq 1}, m \in \mathbb{Z}. \tag{13}$$

Let $\ker\chi \subset U(L\mathfrak{g}_{\geq 1})$ be the kernel of the algebra homomorphism $\chi : U(L\mathfrak{g}_{\geq 1}) \to \mathbb{C}$. Define $I_\chi \underset{\text{def}}{=} U(L\mathfrak{g}_{>0})\ker\chi$. Then, I_χ is a two-sided ideal of $U(L\mathfrak{g}_{>0})$. Next, define

$$N(\chi) \underset{\text{def}}{=} U(L\mathfrak{g}_{>0})/I_\chi. \tag{14}$$

Now, let $\Phi_u(n)$, with $u \in \mathfrak{g}_{>0}$ and $n \in \mathbb{Z}$, denote the image of $u(n) \in L\mathfrak{g}_{>0}$ in the algebra $N(\chi)$. With a slight abuse of notation, we write $\Phi_{u_\alpha}(n)$ as $\Phi_\alpha(n)$ for $\alpha \in \Delta_{\frac{1}{2}}$ and $n \in \mathbb{Z}$. Then, the superalgebra $N(\chi)$ is generated by $\Phi_\alpha(n)$, where $\alpha \in \Delta_{\frac{1}{2}}$ and $n \in \mathbb{Z}$, with the relations

$$[\Phi_\alpha(m), \Phi_\beta(n)] = \langle u_\alpha | u_\beta \rangle_{\text{ne}} \delta_{m+n,-1} \quad \text{for } \alpha, \beta \in \Delta_{\frac{1}{2}}, m, n \in \mathbb{Z}. \tag{15}$$

The elements $\{\Phi_\alpha(n)\}$ are called the *neutral free superfermions* (see Example 1.2 of Ref. [17]).

Let $\{u^\alpha\}_{\alpha\in\Delta_{\frac{1}{2}}}$ be the basis of $\mathfrak{g}_{\frac{1}{2}}$ dual to $\{u_\alpha\}_{\alpha\in\Delta_{\frac{1}{2}}}$ with respect to $\langle\ |\ \rangle_{\text{ne}}$, that is, $\langle u_\alpha | u^\beta \rangle_{\text{ne}} = \delta_{\alpha,\beta}$. We set $\Phi^\alpha(n) = \Phi_{u^\alpha}(n)$ for $\alpha \in \Delta_{\frac{1}{2}}$ and $n \in \mathbb{Z}$, and thus,

$$[\Phi_\alpha(m), \Phi^\beta(n)] = \delta_{\alpha,\beta}\delta_{m+n,-1} \quad \text{with } \alpha, \beta \in \Delta_{\frac{1}{2}},\ m, n \in \mathbb{Z}. \tag{16}$$

3.2. Let $\mathcal{F}^{\text{ne}}(\chi)$ be the irreducible representations of $N(\chi)$ generated by the vector $\mathbf{1}_\chi$ with the property

$$\Phi_\alpha(n)\mathbf{1}_\chi = 0 \quad \text{for } \alpha \in \Delta_{\frac{1}{2}} \text{ and } n \geq 0. \tag{17}$$

Note that $\mathcal{F}^{\text{ne}}(\chi)$ is naturally a $L\mathfrak{g}_{>0}$-module through the algebra homomorphism $L\mathfrak{g}_{>0} \ni u(m) \mapsto \Phi_u(m) \in N(\chi)$.

There is a unique semisimple action of $\widehat{\mathfrak{h}}$ on $\mathcal{F}^{\text{ne}}(\chi)$ such that the following hold:

$$\begin{aligned}
&h\mathbf{1}_\chi = 0 \quad \text{for } h \in \mathfrak{h},\\
&\Phi_\alpha(n)\mathcal{F}^{\text{ne}}(\chi)^\lambda \subset \mathcal{F}^{\text{ne}}(\chi)^{\lambda+\alpha+n\delta} \quad \text{for } \alpha \in \Delta_{\frac{1}{2}}, n \leq -1 \text{ and } \lambda \in \widehat{\mathfrak{h}}^*.
\end{aligned}$$

Lemma 3.2.1. *We have* $\Phi_\alpha(n)\mathcal{F}^{\text{ne}}(\chi)^\lambda \subset \sum_{\substack{\beta\in\Delta_{\frac{1}{2}}\\ \bar{\chi}([u_\beta,u_\alpha])\neq 0}} \mathcal{F}^{\text{ne}}(\chi)^{\lambda-\beta+(n+1)\delta}$ *for* $\alpha \in \Delta_{\frac{1}{2}}$, $n \geq 0$ *and* $\lambda \in \widehat{\mathfrak{h}}^*$.

Proof. By definition we have

$$\Phi^{\alpha}(n)\mathcal{F}^{\mathrm{ne}}(\chi)^{\lambda} \subset \mathcal{F}^{\mathrm{ne}}(\chi)^{\lambda-\alpha+(n+1)\delta} \text{ for } n \geq 0 \tag{18}$$

(see (16)). But $\Phi_{\alpha}(n) = \sum_{\beta\in\Delta_{\frac{1}{2}}} \bar{\chi}([u_{\beta}, u_{\alpha}])\Phi^{\beta}(n)$ for $\alpha \in \Delta_{\frac{1}{2}}$. □

3.3. Let $\mathcal{C}l(L\mathfrak{g}_{>0})$ be the *Clifford superalgebra* (or the *charged free superfermions*) associated with $L\mathfrak{g}_{>0} \oplus (L\mathfrak{g}_{>0})^{*}$ and its natural bilinear from. The superalgebra $\mathcal{C}l(L\mathfrak{g}_{>0})$ is generated by $\psi_{\alpha}(n)$ and $\psi^{\alpha}(n)$, where $\alpha \in \Delta_{>0}$ and $n \in \mathbb{Z}$, with the relations

$$[\psi_{\alpha}(m), \psi^{\beta}(n)] = \delta_{\alpha,\beta}\delta_{m+n,0},$$
$$[\psi_{\alpha}(m), \psi_{\beta}(m)] = [\psi^{\alpha}(m), \psi^{\beta}(n)] = 0,$$

where the parity of $\psi_{\alpha}(n)$ and $\psi^{\alpha}(n)$ is reverse to u_{α}.

3.4. Let $\mathcal{F}(L\mathfrak{g}_{>0})$ be the irreducible representation of $\mathcal{C}l(L\mathfrak{g}_{>0})$ generated by the vector $\mathbf{1}$ that satisfies the relations

$$\psi_{\alpha}(n)\mathbf{1} = 0 \quad \text{for } \alpha \in \Delta_{>0} \text{ and } n \geq 0,$$
$$\text{and } \psi^{\alpha}(n)\mathbf{1} = 0 \quad \text{for } \alpha \in \Delta_{>0} \text{ and } n > 0.$$

The space $\mathcal{F}(L\mathfrak{g}_{>0})$ is graded; that is, $\mathcal{F}(L\mathfrak{g}_{>0}) = \bigoplus_{i\in\mathbb{Z}} \mathcal{F}^{i}(L\mathfrak{g}_{>0})$, where the degree is determined from the assignments $\deg \mathbf{1} = 0$, $\deg \psi_{\alpha}(n) = -1$ and $\deg \psi^{\alpha}(n) = 1$, with $\alpha \in \Delta_{>0}$ and $n \in \mathbb{Z}$.

There is a natural semisimple action of $\widehat{\mathfrak{h}}$ on $\mathcal{F}(L\mathfrak{g}_{>0})$, namely, $\mathcal{F}(L\mathfrak{g}_{>0}) = \bigoplus_{\lambda\in\widehat{\mathfrak{h}}^{*}} \mathcal{F}(L\mathfrak{g}_{>0})^{\lambda}$. This is defined by the relations $h\mathbf{1} = 0$ ($h \in \widehat{\mathfrak{h}}$), $\psi_{\alpha}(n)\mathcal{F}(L\mathfrak{g}_{>0})^{\lambda} \subset \mathcal{F}(L\mathfrak{g}_{>0})^{\lambda+\alpha+n\delta}$, $\psi^{\alpha}(n)\mathcal{F}(L\mathfrak{g}_{>0})^{\lambda} \subset \mathcal{F}(L\mathfrak{g}_{>0})^{\lambda-\alpha+n\delta}$, where $\alpha \in \Delta_{>0}$ and $n \in \mathbb{Z}$.

3.5. For $V \in Obj\mathcal{O}_{k}$, define

$$C(V) \underset{\mathrm{def}}{=} V \otimes \mathcal{F}^{\mathrm{ne}}(\chi) \otimes \mathcal{F}(L\mathfrak{g}_{>0}) = \sum_{i\in\mathbb{Z}} C^{i}(V), \tag{19}$$

where $C^{i}(V) \underset{\mathrm{def}}{=} V \otimes \mathcal{F}^{\mathrm{ne}}(\chi) \otimes \mathcal{F}^{i}(L\mathfrak{g}_{>0})$.

Let $\widehat{\mathfrak{h}}$ act on $C(V)$ by the tensor product action. Then, we have $C(V) = \bigoplus_{\lambda\in\widehat{\mathfrak{h}}^{*}} C(V)^{\lambda}$, where $C(V)^{\lambda} = \sum_{\mu_1+\mu_2+\mu_3=\lambda} V^{\mu_1} \otimes \mathcal{F}^{\mathrm{ne}}(\chi)^{\mu_2} \otimes \mathcal{F}(L\mathfrak{g}_{>0})^{\mu_3}$. Note that we also have

$$C(V) = \bigoplus_{\mu\leq\lambda} C(V)^{\mu} \text{ and } \dim_{\mathbb{C}} C(V)^{\mu} < \infty \text{ for all } \mu \in \widehat{\mathfrak{h}}^{*} \tag{20}$$

with an object V of $\mathcal{O}_{k}^{\leq\lambda}$.

3.6. Define the odd operator d on $C(V)$ by

$$\begin{aligned} d \underset{\mathrm{def}}{=} & \sum_{\substack{\alpha\in\Delta_{>0} \\ n\in\mathbb{Z}}} (-1)^{p(\alpha)}(u_{\alpha}(-n) + \Phi_{u_{\alpha}}(-n))\psi^{\alpha}(n) \\ & - \frac{1}{2} \sum_{\substack{\alpha,\beta,\gamma\in\Delta_{>0} \\ k+l+m=0}} (-1)^{p(\alpha)p(\gamma)}([u_{\alpha}, u_{\beta}]|u_{-\gamma})\psi^{\alpha}(k)\psi^{\beta}(l)\psi_{\gamma}(m). \end{aligned} \tag{21}$$

Then, we have

$$d^{2} = 0, \quad dC^{i}(V) \subset C^{i+1}(V). \tag{22}$$

Thus, $(C(V), d)$ is a cohomology complex. Define

$$H^i(V) \underset{\text{def}}{=} H^i(C(V), d) \text{ with } i \in \mathbb{Z}. \tag{23}$$

Remark 3.6.1. By definition, we have

$$H^\bullet(V) = H^{\frac{\infty}{2}+\bullet}(L\mathfrak{g}_{>0}, V \otimes \mathcal{F}^{\mathrm{ne}}(\chi)),$$

where $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{g}_{>0}, V)$ is the semi-infinite cohomology [9] of the Lie superalgebra $L\mathfrak{g}_{>0}$ with coefficients in V.

3.7. Decompose d as $d = d^\chi + d^{\mathrm{st}}$, where

$$d^\chi \underset{\text{def}}{=} \sum_{\substack{\alpha\in\Delta_{\frac{1}{2}} \\ n\geq 0}} (-1)^{p(\alpha)} \Phi_\alpha(n)\psi^\alpha(-n) + \sum_{\alpha\in\Delta_1} (-1)^{p(\alpha)} \chi(u_\alpha(-1))\psi^\alpha(1) \tag{24}$$

and $d^{\mathrm{st}} \underset{\text{def}}{=} d - d^\chi$. Then, by Lemma 3.2.1, we have

$$d^\chi C(V)^\lambda \subset \sum_{\substack{\alpha\in\Delta_1 \\ \bar{\chi}(u_\alpha)\neq 0}} C(V)^{\lambda-\alpha+\delta}, \quad d^{\mathrm{st}} C(V)^\lambda \subset C(V)^\lambda \tag{25}$$

for all λ. Therefore, by (22), it follows that

$$(d^\chi)^2 = (d^{\mathrm{st}})^2 = \{d^\chi, d^{\mathrm{st}}\} = 0. \tag{26}$$

Remark 3.7.1. We have

$$H^\bullet(C(V), d^{\mathrm{st}}) = H^{\frac{\infty}{2}+\bullet}(L\mathfrak{g}_{>0}, V \otimes \mathcal{F}^{\mathrm{ne}}(\chi_0)), \tag{27}$$

where $\mathcal{F}^{\mathrm{ne}}(\chi_0)$ is the $L\mathfrak{g}_{>0}$-module associated with the trivial character χ_0 of $L\mathfrak{g}_{\geq 1}$ defined similarly to $\mathcal{F}^{\mathrm{ne}}(\chi)$.

3.8. Define

$$\mathbf{D}^{\mathcal{W}} \underset{\text{def}}{=} x + \mathbf{D} \in \widehat{\mathfrak{h}}, \tag{28}$$

Here, x is the semisimple element in the $\mathfrak{sl}_2$-triple, as in Subsection 2.1. Set

$$\widehat{\mathfrak{t}} \underset{\text{def}}{=} \mathfrak{h}^f \oplus \mathbb{C}\mathbf{D}^{\mathcal{W}} \subset \widehat{\mathfrak{h}}. \tag{29}$$

Let $\widehat{\mathfrak{t}}^*$ be the dual of $\widehat{\mathfrak{t}}$. For $\lambda \in \widehat{\mathfrak{h}}^*$, let $\xi_\lambda \in \widehat{\mathfrak{t}}^*$ denote its restriction to $\widehat{\mathfrak{t}}$.

3.9. Let $V \in Obj\mathcal{O}_k$ and let

$$C(V) = \bigoplus_{\xi\in\widehat{\mathfrak{t}}^*} C(V)_\xi, \quad C(V)_\xi = \sum_{\substack{\lambda\in\widehat{\mathfrak{h}}^* \\ \xi_\lambda = \xi}} C(V)^\lambda \tag{30}$$

be the weight space decomposition with respect to the action of $\widehat{\mathfrak{t}} \subset \widehat{\mathfrak{h}}$. Here and throughout, we set

$$M_\xi \underset{\text{def}}{=} \{m \in M \mid tm = \langle \xi, t\rangle m \ (\forall t \in \widehat{\mathfrak{t}})\}$$

for a $\mathfrak{t}$-module M. By (25), we see that

$$dC(V)_\xi \subset C(V)_\xi \text{ for any } \xi \in \widehat{\mathfrak{t}}^*.$$

Hence, the cohomology space $H^\bullet(V)$ decomposes as

$$H^\bullet(V) = \bigoplus_{\xi\in\widehat{\mathfrak{t}}^*} H^\bullet(V)_\xi, \quad H^\bullet(V)_\xi = H^\bullet(C(V)_\xi, d). \tag{31}$$

Note that the weight space $C(V)_\xi$, with $\xi \in \widehat{\mathfrak{t}}^*$, is not finite dimensional in general because $[\widehat{\mathfrak{t}}, e(-1)] = 0$.

Remark 3.9.1. As discussed in Remark 5.3.1, the operator $\mathbf{D}^{\mathcal{W}}$ is essentially $-L(0)$, where $L(0)$ is the zero-mode of the Virasoro field, provided that $k \neq -h^\vee$.

4. The assumption $f = f_\theta$

4.1. The gradation (2) is called *minimal* if

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\frac{1}{2}} \oplus \mathfrak{g}_1, \quad \mathfrak{g}_{-1} = \mathbb{C}f \text{ and } \quad \mathfrak{g}_1 = \mathbb{C}e. \tag{32}$$

As shown in Section 5 of Ref. [19], in this case, one can choose a root system of $\mathfrak{g}$ so that $e = e_\theta$ and $f = f_\theta$, which are the roots vectors attached to θ and $-\theta$, where θ is the corresponding highest root.

The condition (32) simplifies the theory in many ways. *In this section we assume that $f = f_\theta$ and the condition* (32) *is satisfied.* Also, we normalize $(\ |\ )$ as $(\theta|\theta) = 2$.

4.2. From the $\mathfrak{sl}_2$-representation theory, we have

$$\mathfrak{g}^f = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{g}_0^f\ , \tag{33}$$

$$\mathfrak{g}_0^f = \mathfrak{n}_{0,-} \oplus \mathfrak{h}^f \oplus \mathfrak{n}_{0,+}. \tag{34}$$

In particular,

$$\mathfrak{h} = \mathfrak{h}^f \oplus \mathbb{C}x, \quad \mathfrak{n}_- \subset \mathfrak{g}^f, \tag{35}$$

and we have the exact sequence

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathbb{C}\alpha_0 \oplus \mathbb{C}\Lambda_0 & \hookrightarrow & \widehat{\mathfrak{h}}^* & \rightarrow & \widehat{\mathfrak{t}}^* & \rightarrow & 0 \\ & & & & \lambda & \mapsto & \xi_\lambda. & & \end{array} \tag{36}$$

Here, $\alpha_0 = \delta - \theta$. Therefore, for $\lambda, \mu \in \widehat{\mathfrak{h}}_k^*$,

$$\xi_\lambda = \xi_\mu \text{ if and only if } \lambda \equiv \mu \pmod{\alpha_0}. \tag{37}$$

Let $\widehat{Q}_+^{\mathfrak{t}} \subset \widehat{\mathfrak{t}}^*$ be the image of $\widehat{Q}_+ \subset \widehat{\mathfrak{h}}^*$ in $\widehat{\mathfrak{t}}^*$. Then, by (36) we have

$$\langle \eta, \mathbf{D}^{\mathcal{W}} \rangle \geq 0 \quad \text{for all } \eta \in \widehat{Q}_+^{\mathfrak{t}}. \tag{38}$$

Define a partial ordering on $\widehat{\mathfrak{t}}^*$ by $\xi \leq \xi' \iff \xi' - \xi \in \widehat{Q}_+^{\mathfrak{t}}$. Then, for $\lambda, \mu \in \widehat{\mathfrak{h}}^*$, we have the property

$$\xi_\lambda \leq \xi_\mu \text{ if } \lambda \leq \mu. \tag{39}$$

In particular,

$$V = \bigoplus_{\xi \leq \xi_\lambda} V_\xi \quad \text{for an object } V \text{ of } \mathcal{O}_k^{\leq \lambda}. \tag{40}$$

4.3. Let $\widehat{\mathfrak{g}} = \bigoplus_{\eta \in \widehat{\mathfrak{t}}^*} (\widehat{\mathfrak{g}})_\eta$ be the weight space decomposition with respect to the adjoint action of $\widehat{\mathfrak{t}}$. Then, we have

$$(\widehat{\mathfrak{g}})_0 = \widehat{\mathfrak{h}} \oplus \mathbb{C}e(-1) \oplus \mathbb{C}f(1) \tag{41}$$

(recall that $e = e_\theta$ and $f = f_\theta$).

Let $V \in \mathcal{O}_k$. Then, each weight space V_ξ, where $\xi \in \widehat{\mathfrak{t}}^*$, is a module over $(\widehat{\mathfrak{g}})_0$. Also, we have

$$V_{\xi_\lambda} = \sum_{\substack{\mu \in \widehat{\mathfrak{h}}_k^* \\ \mu \equiv \lambda \pmod{\alpha_0}}} V^\mu \text{ for } \lambda \in \widehat{\mathfrak{h}}_k^*. \tag{42}$$

Let $(\mathfrak{sl}_2)_0 \cong \mathfrak{sl}_2$ denote the subalgebra of $(\widehat{\mathfrak{g}})_0$ generated by $e(-1)$ and $f(1)$.

4.4. Let $\mathcal{O}(\mathfrak{sl}_2)$ be the Bernstein-Gel'fand-Gel'fand category [3] of the Lie algebra $\mathfrak{sl}_2 = \langle e, x, f \rangle$ (defined by the commutation relations (1)). That is, $\mathcal{O}(\mathfrak{sl}_2)$ is the full subcategory of the category of left $\mathfrak{sl}_2$-modules consisting of modules V such that (1) V is finitely generated over $\mathfrak{sl}_2$, (2) e is locally nilpotent on V, (3) x acts semisimply on V.

Let $\dot{\mathcal{O}}_k$ be the full subcategory of $\mathcal{O}_k$ consisting of objects V such that each V_ξ, with $\xi \in \widehat{\mathfrak{t}}^*$, belongs to $\mathcal{O}(\mathfrak{sl}_2)$ (viewed as a module over $(\mathfrak{sl}_2)_0 \cong \mathfrak{sl}_2$). It is clear that $\dot{\mathcal{O}}_k$ is abelian.

Lemma 4.4.1.

(1) *Any Verma module $M(\lambda)$, with $\lambda \in \widehat{\mathfrak{h}}_k^*$, belongs to $\dot{\mathcal{O}}_k$.*
(2) *Any simple module $L(\lambda)$, with $\lambda \in \widehat{\mathfrak{h}}_k^*$, belongs to $\dot{\mathcal{O}}_k$.*
(3) *Any object of $\mathcal{O}_k^{\triangle}$ belongs to $\dot{\mathcal{O}}_k$.*
(4) *Any object of $\mathcal{O}_k^{\triangledown}$ belongs to $\dot{\mathcal{O}}_k$.*

Proof. (1) Certainly, on $M(\lambda)$, $f(1)$ is locally nilpotent and $h_0 = [f(1), e(-1)]$ acts semisimply. We have to show that each $M(\lambda)_\xi$, with $\xi \in \widehat{\mathfrak{t}}^*$, is finitely generated over $(\mathfrak{sl}_2)_0$. Clearly, this follows from (37), (39) and the PBW theorem. (2), (3) These assertions follow from the first assertion. (4) The category $\mathcal{O}(\mathfrak{sl}_2)$ is closed under the operation of taking (graded) dual. Hence, $\dot{\mathcal{O}}_k$ is closed under the operation of taking (graded) dual. Therefore (4) follows from the third assertion. □

Let $\dot{\mathcal{O}}_k^{\leq\lambda}$ be the full subcategory of $\mathcal{O}_k$ consisting of objects that belong to both $\dot{\mathcal{O}}_k$ and $\mathcal{O}_k^{\leq\lambda}$. Then, by Lemma 4.4.1, $P_{\leq\lambda}(\mu)$ and $I_{\leq\lambda}(\mu)$ ($\mu \leq \lambda$) belong to $\dot{\mathcal{O}}_k^{\leq\lambda}$.

The following proposition asserts that every object of $\mathcal{O}_k$ can be obtained as an injective limit of objects of $\dot{\mathcal{O}}_k$.

Proposition 4.4.2. *Let V be an object of $\mathcal{O}_k$. Then, there exits a sequence $V_1 \subset V_2 \subset V_3 \dots$ of objects of $\dot{\mathcal{O}}_k$ such that $V = \bigcup_i V_i$.*

Proof. Note that, because each projective module $P_{\leq\lambda}(\mu)$ belongs to $\dot{\mathcal{O}}_k$, so too do finitely generated objects. Let $\{0\} = V_0 \subset V_1 \subset V_2 \subset V_3 \dots$ be a highest weight filtration of V, so that $V = \bigcup_i V_i$, and each successive subquotient V_i/V_{i-1} is a highest weight module. In particular, each V_i is finitely generated, and hence belongs to $\dot{\mathcal{O}}_k$. □

Lemma 4.4.3. *Let $\xi \in \widehat{\mathfrak{t}}^*$. Then, for any object V of $\dot{\mathcal{O}}_k^{\leq\lambda}$, with $\lambda \in \widehat{\mathfrak{h}}_k^*$, there exits a finitely generated submodule V' of V such that $(V/V')_{\xi'} = \{0\}$ if $\xi' \geq \xi$, where $\xi' \in \widehat{\mathfrak{t}}^*$.*

Proof. Let $\mathcal{P} = \{v_1, v_2, \dots\}$ be a set of generators of V such that (1) each v_i belongs to V^{μ_i} for some $\mu_i \in \widehat{\mathfrak{h}}^*$, and (2) if we set $V_i = \sum_{r=1}^{i} U(\widehat{\mathfrak{g}})v_r$ (and $V_0 = \{0\}$), then each successive subquotient V_i/V_{i-1} is a (nonzero) highest weight module with highest weight μ_i. (Therefore, $V_1 \subset V_2 \subset \dots$ is a highest weight filtration of V.) Then, by definition, we have $\sharp\{j \geq 1 \mid \mu_j = \mu\} \leq [V : L(\mu)]$ for $\mu \in \widehat{\mathfrak{h}}^*$. Next, let $\mathcal{P}_{\geq \xi} = \{v_j \in \mathcal{P} \mid \xi_{\mu_j} \geq \xi\} \subset \mathcal{P}$. Then, by the definition of $\dot{\mathcal{O}}_k$, $\mathcal{P}_{\geq\xi}$ is a finite subset of $\mathcal{P}$. The assertion follows, because $V' = \sum_{v \in \mathcal{P}_{\geq\xi}} U(\widehat{\mathfrak{g}})v \subset V$ satisfies the desired properties. □

4.5. Observe that we have the following:

$$\mathcal{F}^{\mathrm{ne}}(\chi) = \bigoplus_{\xi \leq 0} \mathcal{F}^{\mathrm{ne}}(\chi)_\xi, \quad \dim_{\mathbb{C}} \mathcal{F}^{\mathrm{ne}}(\chi)_\xi < \infty \ (\forall \xi), \quad \mathcal{F}^{\mathrm{ne}}(\chi)_0 = \mathbb{C}\mathbf{1}_\chi, \tag{43}$$

$$\mathcal{F}(L\mathfrak{g}_{>0}) = \bigoplus_{\xi \leq 0} \mathcal{F}(L\mathfrak{g}_{>0})_\xi, \quad \mathcal{F}(L(\mathfrak{g}_{>0}))_0 = \mathbb{C}\mathbf{1} \oplus \mathbb{C}\psi_\theta(-1)\mathbf{1} \tag{44}$$

Thus, if $V = \bigoplus_{\xi' \leq \xi} V_{\xi'}$ for some $\xi' \in \widehat{\mathfrak{t}}^*$, then $C(V) = \bigoplus_{\xi' \leq \xi} C(V)_{\xi'}$. Hence we have the following assertion.

Lemma 4.5.1. *Let V be an object of $\mathcal{O}_k$. Suppose that $V = \bigoplus_{\substack{\xi' \in \widehat{\mathfrak{t}}^* \\ \xi' \leq \xi}} V_{\xi'}$ for some $\xi \in \widehat{\mathfrak{t}}^*$. Then, $H^\bullet(V) = \bigoplus_{\xi' \leq \xi} H^\bullet(V)_{\xi'}$. In particular, $H^\bullet(V) = \bigoplus_{\xi \leq \xi_\lambda} H^\bullet(V)_\xi$ for $V \in Obj\mathcal{O}_k^{\leq\lambda}$.*

Lemma 4.5.2. *Let $\xi \in \widehat{\mathfrak{t}}^*$. Then, for any object V of $\dot{\mathcal{O}}_k^{\leq\lambda}$, where $\lambda \in \widehat{\mathfrak{h}}_k^*$, there exits a finitely generated submodule V' of V such that $H^\bullet(V)_\xi = H^\bullet(V')_\xi$.*

Proof. Let V' be as in Lemma 4.4.3. Then, from the exact sequence $0 \to V' \to V \to V/V' \to 0$, we obtain the long exact sequence

$$\cdots \to H^{i-1}(V/V') \to H^i(V') \to H^i(V) \to H^i(V/V') \to \dots. \tag{45}$$

Clearly, the restriction of (45) to the weight space ξ remains exact. The desired result then follows from Lemma 4.4.3 and Lemma 4.5.1. □

4.6. Here and throughout, we identify $\mathcal{F}(L(\mathfrak{g}_{>0}))_0$ with the exterior power module $\Lambda(\mathbb{C}e(-1))$ by identifying $\psi_\theta(-1)$ with $e(-1)$ (see (44)). Let $\mathbb{C}_\chi$ be the one-dimensional module over the commutative Lie algebra $\mathbb{C}e(-1)$ defined by the character $\chi_{|\mathbb{C}e(-1)}$, that is, the one-dimensional $\mathbb{C}e(-1)$-module on which $e(-1)$ acts as the identity operator. Also, let V be an object of $\mathcal{O}_k$. Then, $V_\xi \otimes \mathbb{C}_\chi$, where $\xi \in \widehat{\mathfrak{t}}^*$, is a module over $\mathbb{C}e(-1)$ by the tensor product action.

Lemma 4.6.1. *Let $V \in Obj\mathcal{O}_k^{\leq\lambda}$, with $\lambda \in \widehat{\mathfrak{h}}_k^*$. Then, we have*

$$H^i(V)_{\xi_\lambda} = \begin{cases} H_{-i}(\mathbb{C}e(-1), V_{\xi_\lambda} \otimes \mathbb{C}_\chi) & (i = 0, -1), \\ \{0\} & (\mathit{otherwise}). \end{cases}$$

Proof. Because V is an object of $\mathcal{O}_k^{\leq\lambda}$, we have

$$C(V)_{\xi_\lambda} = V_{\xi_\lambda} \otimes \mathcal{F}(L(\mathfrak{g}_{>0}))_0 (= V_{\xi_\lambda} \otimes \Lambda(\mathbb{C}e(-1))).$$

Next, observe that

$$d_{|C(V)_{\xi_\lambda}} = \bar{d} = e(-1)\psi^\theta(1) + \psi^\theta(1).$$

From this, it follows that, for $V \in Obj\mathcal{O}_k^{\leq\lambda}$, the subcomplex $(C(V)_{\xi_\lambda}, d)$ is identically the Chevalley complex for calculating the (usual) Lie algebra homology $H_\bullet(\mathbb{C}e(-1), V_{\xi_\lambda} \otimes \mathbb{C}_\chi)$ (with the opposite grading). $\square$

4.7. Recall $\mathfrak{sl}_2 = \langle e, x, f\rangle$. Let $\bar{M}_{\mathfrak{sl}_2}(a) \in Obj\mathcal{O}(\mathfrak{sl}_2)$ be the Verma module of $\mathfrak{sl}_2$ with highest weight $a \in \mathbb{C}$ and $\bar{L}_{\mathfrak{sl}_2}(a)$ be its unique simple quotient. Here, the highest weight is the largest eigenvalue of $2x$ (see (1)).

Let $\mathbb{C}_{\bar{\chi}_-}$ be the one-dimensional $\mathbb{C}f$-module on which f acts as the identity operator.

Proposition 4.7.1.

(1) *For* $a \in \mathbb{C}$, $H_i(\mathbb{C}f, \bar{M}_{\mathfrak{sl}_2}(a) \otimes \mathbb{C}_{\bar{\chi}_-}) = \begin{cases} \mathbb{C} & (i=0), \\ \{0\} & (i=1). \end{cases}$

(2) *For* $a \in \mathbb{C}$, $H_i(\mathbb{C}f, \bar{L}_{\mathfrak{sl}_2}(a) \otimes \mathbb{C}_{\bar{\chi}_-}) = \begin{cases} \mathbb{C} & (i=0 \text{ and } a \notin \{0,1,2,\dots\}), \\ \{0\} & (\text{otherwise}). \end{cases}$

(3) *For* $a \in \mathbb{C}$, $H_i(\mathbb{C}f, \bar{M}_{\mathfrak{sl}_2}(a)^* \otimes \mathbb{C}_{\bar{\chi}_-}) = \begin{cases} \mathbb{C} & (i=0), \\ \{0\} & (i=1). \end{cases}$

(4) *For any object* V *of* $\mathcal{O}(\mathfrak{sl}_2)$, *we have* $H_1(\mathbb{C}f, V \otimes \mathbb{C}_{\bar{\chi}_-}) = \{0\}$.

(5) *For any object* V *of* $\mathcal{O}(\mathfrak{sl}_2)$, *we have* $\dim_{\mathbb{C}} H_0(\mathbb{C}f, V \otimes \mathbb{C}_{\bar{\chi}_-}) < \infty$.

Proof. (1) Since $\bar{M}_{\mathfrak{sl}_2}(a)$ is free over $\mathbb{C}f$, so is $\bar{M}_{\mathfrak{sl}_2}(a) \otimes \mathbb{C}_{\bar{\chi}_-}$. (2) The case in which $a \notin \{0,1,2,\dots\}$ follows from the first assertion. Otherwise, $\bar{L}_{\mathfrak{sl}_2}(a)$ is finite dimensional. Hence, f is nilpotent on $\bar{L}_{\mathfrak{sl}_2}(a)$. But this implies that the corresponding Chevalley complex is acyclic, by the argument of Theorem 2.3 of Ref. [13]. (3) The case in which $a \notin \{0,1,2,\dots\}$ follows from the first assertion. Otherwise, we have the following exact sequence in $\mathcal{O}(\mathfrak{sl}_2)$:

$$0 \to \bar{L}_{\mathfrak{sl}_2}(a) \to \bar{M}_{\mathfrak{sl}_2}(a)^* \to \bar{M}_{\mathfrak{sl}_2}(-a-2) \to 0.$$

This induces the exact sequence

$$0 \to \bar{L}_{\mathfrak{sl}_2}(a) \otimes \mathbb{C}_{\bar{\chi}_-} \to \bar{M}_{\mathfrak{sl}_2}(a)^* \otimes \mathbb{C}_{\bar{\chi}_-} \to \bar{M}_{\mathfrak{sl}_2}(-a-2) \otimes \mathbb{C}_{\bar{\chi}_-} \to 0.$$

Hence, the assertion is obtained from the first and the second assertions by considering the corresponding long exact sequence of the Lie algebra homology. (4) Recall that $\mathcal{O}(\mathfrak{sl}_2)$ has enough injectives, and each injective object I admits a finite filtration such that each successive quotient is isomorphic to $\bar{M}_{\mathfrak{sl}_2}(a)^*$ for some $a \in \mathbb{C}$. Therefore, the third assertion implies that $H_1(\mathbb{C}f, I \otimes \mathbb{C}_{\bar{\chi}_-}) = \{0\}$ for any injective object I in $\mathcal{O}(\mathfrak{sl}_2)$ (cf. Theorem 8.2 of [1]). For a given $V \in Obj\mathcal{O}(\mathfrak{sl}_2)$, let $0 \to V \to I \to V/I \to 0$ be an exact sequence in $\mathcal{O}(\mathfrak{sl}_2)$ such that I is injective. Then, from the associated long exact sequence, it is proved that $H_1(\mathbb{C}f, V \otimes \mathbb{C}_{\bar{\chi}_-}) = \{0\}$. (4) By the third assertion the correspondence $V \mapsto H_0(\mathbb{C}f, V \otimes \mathbb{C}_{\bar{\chi}_-})$ defines an exact functor from $\mathcal{O}(\mathfrak{sl}_2)$ to the category of $\mathbb{C}$-vector spaces. Because the assertion follows from the first assertion for Verma modules, it also holds for any projective object P of $\mathcal{O}(\mathfrak{sl}_2)$, since P has a (finite) Verma flag. This completes the proof, as $\mathcal{O}(\mathfrak{sl}_2)$ has enough projectives. $\square$

4.8. For $\lambda \in \widehat{\mathfrak{h}}^*$, let

$$|\lambda\rangle = v_\lambda \otimes \mathbf{1} \in C(M(\lambda)).$$

Then, $d|\lambda\rangle = 0$, and thus $|\lambda\rangle$ defines an element of $H^0(M(\lambda))$. Again with a slight abuse of notation, we denote the image of $|\lambda\rangle$ under the natural map $C(M(\lambda)) \to C(L(\lambda))$ by $|\lambda\rangle$. Also, let

$$|\lambda\rangle^* = v_\lambda^* \otimes \mathbf{1} \in H^0(M(\lambda)^*), \tag{46}$$

where v_λ^* is the vector of $M(\lambda)^*$ dual to v_λ.

Proposition 4.8.1. *For any $\lambda \in \widehat{\mathfrak{h}}^*$, we have the following:*

(1) $H^i(M(\lambda))_{\xi_\lambda} = \begin{cases} \mathbb{C}|\lambda\rangle & (\textit{if } i = 0), \\ \{0\} & (\textit{otherwise}). \end{cases}$

(2) $H^i(L(\lambda))_{\xi_\lambda} = \begin{cases} \mathbb{C}|\lambda\rangle & (\textit{if } i = 0 \textit{ and } \langle \lambda, \alpha_0^\vee \rangle \notin \{0, 1, 2, \dots\}), \\ \{0\} & (\textit{otherwise}). \end{cases}$

(3) $H^i(M(\lambda)^*)_{\xi_\lambda} = \begin{cases} \mathbb{C}|\lambda\rangle^* & (\textit{if } i = 0), \\ \{0\} & (\textit{otherwise}). \end{cases}$

Proof. Observe that $M(\lambda)_{\xi_\lambda}$ is isomorphic to $\bar{M}_{\mathfrak{sl}_2}(\langle \lambda, \alpha_0^\vee \rangle)$ as a module over $(\mathfrak{sl}_2)_0$. Similarly, $L(\lambda)_{\xi_\lambda}$ and $M(\lambda)^*_{\xi_\lambda}$ are isomorphic to $\bar{L}_{\mathfrak{sl}_2}(\langle \lambda, \alpha_0^\vee \rangle)$ and $\bar{M}_{\mathfrak{sl}_2}(\langle \lambda, \alpha_0^\vee \rangle)^*$, respectively. Hence, the assertion follows from Lemma 4.6.1 and Proposition 4.7.1. □

4.9. Let V be an object of $\dot{\mathcal{O}}_k$. For a given $\xi \in \widehat{\mathfrak{t}}^*$, consider the (usual) Lie algebra homology $H_\bullet(\mathbb{C}e(-1), V_\xi \otimes \mathbb{C}_\chi)$. It is calculated using the Chevalley complex $(V_\xi \otimes \Lambda(\mathbb{C}e(-1)), \bar{d})$, (see Section 4.6). The action of $\widehat{\mathfrak{t}}$ on V_ξ commutes with $\bar{d}$. Thus, there is a natural action of $\widehat{\mathfrak{t}}$ on $H_\bullet(\mathbb{C}e(-1), V \otimes \mathbb{C}_\chi)$;

$$H_\bullet(\mathbb{C}e(-1), V \otimes \mathbb{C}_\chi) = \bigoplus_{\xi \in \widehat{\mathfrak{t}}^*} H_\bullet(\mathbb{C}e(-1), V \otimes \mathbb{C}_\chi)_\xi.$$

By definition, we have

$$H_\bullet(\mathbb{C}e(-1), V \otimes \mathbb{C}_\chi)_\xi = H_\bullet(\mathbb{C}e(-1), V_\xi \otimes \mathbb{C}_\chi) \quad \text{for } \xi \in \widehat{\mathfrak{t}}^*. \tag{47}$$

Hence, from (40) and Proposition 4.7.1, we obtain the following assertion.

Proposition 4.9.1. *Let V be an object of $\dot{\mathcal{O}}_k$. Then, we have the following:*

(1) $H_1(\mathbb{C}e(-1), V \otimes \mathbb{C}_\chi) = \{0\}$.

(2) $H_0(\mathbb{C}e(-1), V \otimes \mathbb{C}_\chi) = \bigoplus_{\xi \leq \xi_\lambda} H_0(\mathbb{C}e(-1), V \otimes \mathbb{C}_\chi)_\xi$ *if $V \in Obj \dot{\mathcal{O}}_k^{\leq \lambda}$ with $\lambda \in \widehat{\mathfrak{h}}^*$.*

(3) *Each weight space $H_0(\mathbb{C}e(-1), V \otimes \mathbb{C}_\chi)_\xi$, $\xi \in \widehat{\mathfrak{t}}^*$, is finite dimensional.*

4.10. We end this section with the following important proposition.

Proposition 4.10.1. *For any object V of $\dot{\mathcal{O}}_k$, each weight space $H^\bullet(V)_\xi$, where $\xi \in \widehat{\mathfrak{t}}^*$, is finite dimensional. Moreover, if $V \in Obj \dot{\mathcal{O}}_k^{\leq \lambda}$, with $\lambda \in \widehat{\mathfrak{h}}^*$, then $H^i(V)_\xi = \{0\}$ if $\frac{1}{2}|i| > \langle \xi_\lambda - \xi, \mathbf{D}^{\mathcal{W}} \rangle$ for $i \in \mathbb{Z}$.*

Proof. We may assume that $V \in Obj \dot{\mathcal{O}}_k^{\leq \lambda}$ for some $\lambda \in \widehat{\mathfrak{h}}_k^*$. Decompose $\mathcal{F}(L\mathfrak{g}_{>0})$ as $\mathcal{F}(L\mathfrak{g}_{>0}) = \mathcal{F}(L\mathfrak{g}_{>0}/\mathbb{C}e(-1)) \otimes \Lambda(\mathbb{C}e(-1))$, where $\mathcal{F}(L(\mathfrak{g}_{>0})/\mathbb{C}e(-1))$ is the subspace of $\mathcal{F}(L(\mathfrak{g}_{>0}))$ spanned by the vectors

$$\psi_{\alpha_1}(m_1) \dots \psi_{\alpha_r}(m_r) \psi^{\beta_1}(n_1) \dots \psi^{\beta_s}(n_s) \mathbf{1},$$

with $\alpha_i, \beta_i \in \Delta_{>0}$, $m_i \leq \begin{cases} -2 & (\text{if } \alpha_i = \theta), \\ -1 & (\text{otherwise}), \end{cases}$ $n_i \leq 0$. Then, we have

$$\mathcal{F}^n(L\mathfrak{g}_{>0}) = \sum_{i-j=n} \mathcal{F}^i(L\mathfrak{g}_{>0}/\mathbb{C}e(-1)) \otimes \Lambda^j(\mathbb{C}e(-1)),$$

where $\mathcal{F}^i(L\mathfrak{g}_{>0}/\mathbb{C}e(-1)) = \mathcal{F}(L\mathfrak{g}_{>0}/\mathbb{C}e(-1)) \cap \mathcal{F}^i(L\mathfrak{g}_{>0})$.

Now, define

$$G^pC^n(V) \underset{\text{def}}{=} V \otimes \mathcal{F}^{\text{ne}}(\chi) \otimes \sum_{\substack{i-j=n \\ i \geq p}} \mathcal{F}^i(L\mathfrak{g}_{>0}/\mathbb{C}e(-1)) \otimes \Lambda^j(\mathbb{C}e(-1)) \subset C^n(V). \tag{48}$$

Then, we have

$$\begin{aligned} &C^n(V) = G^nC^n(V) \supset G^{n+1}C^n(V) \supset G^{n+2}C^n(V) = \{0\}, \\ &dG^pC^n(V) \subset G^pC^{n+1}(V). \end{aligned}$$

The corresponding the spectral sequence, $E_r \Rightarrow H^\bullet(V)$, is the *Hochschild-Serre spectral sequence* (more precisely, the semi-infinite, Lie superalgebra analogue of this spectral sequence) for the ideal $\mathbb{C}e(-1) \subset L\mathfrak{g}_{>0}$. By definition, we have the isomorphism

$$E_1^{p,q} = H_{-q}(\mathbb{C}e(-1), V \otimes \mathcal{F}^{\text{ne}}(\chi)) \otimes \mathcal{F}^p(L\mathfrak{g}_{>0}/\mathbb{C}e(-1)),$$

because the complex $(\sum_p G^pC(V)/G^{p+1}C(V), d)$ is identical to the corresponding Chevalley complex. By Proposition 4.9.1 (1), we have

$$E_1^{p,q} \cong \begin{cases} H_0(\mathbb{C}e(-1), V \otimes \mathbb{C}_\chi) \otimes \mathcal{F}^{\text{ne}}(\chi) \otimes \mathcal{F}^p(L\mathfrak{g}_{>0}/\mathbb{C}e(-1)) & (q = 0) \\ \{0\} & (q \neq 0) \end{cases} \tag{49}$$

as $\widehat{\mathfrak{t}}$-modules for any p.

Next, observe that

$$\begin{aligned} &\mathcal{F}^p(L\mathfrak{g}_{>0}/\mathbb{C}e(-1)) = \bigoplus_{\xi \leq 0} \mathcal{F}^p(L\mathfrak{g}_{>0}/\mathbb{C}e(-1))_\xi, \\ &\dim_{\mathbb{C}} \mathcal{F}^p(L\mathfrak{g}_{>0}/\mathbb{C}e(-1))_\xi < \infty \quad \text{for any } \xi, \\ &\mathcal{F}^p(L\mathfrak{g}_{>0}/\mathbb{C}e(-1))_\xi = \{0\} \text{ unless } \langle \xi, \mathbf{D}^{\mathcal{W}} \rangle \leq -\frac{1}{2}|p|. \end{aligned}$$

Hence, from (43), Proposition 4.9.1 and (49), it follows that

$$E_1^{p,0} = \bigoplus_{\substack{\xi \leq \xi_\lambda \\ \frac{1}{2}|p| \leq \langle \xi_\lambda - \xi, \mathbf{D}^{\mathcal{W}} \rangle}} (E_1^{p,0})_\xi, \quad \dim_{\mathbb{C}}(E_1^{p,0})_\xi < \infty \quad \text{for any } \xi. \tag{50}$$

The assertion is thus proved, as our filtration is compatible with the action of $\widehat{\mathfrak{t}}$. □

5. The Kac-Roan-Wakimoto construction II: the $\mathcal{W}$-algebra construction of superconformal algebras

In this section we recall the definition of the $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ and collect necessary information about its structure.

5.1. Let $V_k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}\otimes\mathbb{C}[t] \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D})} \mathbb{C}_k \in Obj\mathcal{O}_k$ be the universal affine vertex algebra associated with $\mathfrak{g}$ at a given level $k \in \mathbb{C}$. Here, $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}\otimes\mathbb{C}[t] \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D}$ on which $\mathfrak{g}\otimes\mathbb{C}[t] \oplus \mathbb{C}\mathbf{D}$ acts trivially and K acts as $k\,\mathrm{id}$. Hence, $V_k(\mathfrak{g})$ is a quotient of $M(k\Lambda_0)$ as a $\widehat{\mathfrak{g}}$-module. It is known that the space $V_k(\mathfrak{g})$ has a natural vertex (super)algebra structure, and the space

$$C(V_k(\mathfrak{g})) = V_k(\mathfrak{g}) \otimes \mathcal{F}^{\mathrm{ne}}(\chi) \otimes \mathcal{F}(L\mathfrak{g}_{>0}) \tag{51}$$

also has a natural vertex (super)algebra structure (see Ref. [17] for details). Let $|0\rangle = (1\otimes 1)\otimes \mathbf{1}_\chi \otimes \mathbf{1}$ be the canonical vector. Also, let $Y(v,z) \in \mathrm{End}\, C(V_k(\mathfrak{g}))[[z,z^{-1}]]$ be the field corresponding to $v \in C(V_k(\mathfrak{g}))$. Then, by the definition,

$$\begin{aligned}
Y(v(-1)|0\rangle, z) &= v(z) = \sum_{n\in\mathbb{Z}} v(n) z^{-n-1} \quad \text{for } v \in \mathfrak{g},\\
Y(\Phi_\alpha(-1)|0\rangle, z) &= \Phi_\alpha(z) = \sum_{n\in\mathbb{Z}} \Phi_\alpha(n) z^{-n-1} \quad \text{for } \alpha \in \Delta_{\frac{1}{2}},\\
Y(\psi_\alpha(-1)|0\rangle, z) &= \psi_\alpha(z) = \sum_{n\in\mathbb{Z}} \psi_\alpha(n) z^{-n-1} \quad \text{for } \alpha \in \Delta_{>0},\\
Y(\psi_{-\alpha}(0)|0\rangle, z) &= \psi_{-\alpha}(z) = \sum_{n\in\mathbb{Z}} \psi_{-\alpha}(n) z^{-n} \quad \text{for } \alpha \in \Delta_{>0}.
\end{aligned}$$

5.2. Define

$$\mathcal{W}_k(\mathfrak{g}, f) \underset{\mathrm{def}}{=} H^0(V_k(\mathfrak{g})). \tag{52}$$

Then, Y descends to the map

$$Y : \mathcal{W}_k(\mathfrak{g}, f) \to \mathrm{End}\, \mathcal{W}_k(\mathfrak{g}, f)[[z, z^{-1}]], \tag{53}$$

because, as shown in Ref. [17], the following relation holds:

$$[d, Y(v,z)] = Y(dv, z) \quad \text{for all } v \in C(V_k(\mathfrak{g})). \tag{54}$$

Therefore, $\mathcal{W}_k(\mathfrak{g}, f)$ has a vertex algebra structure. The vertex algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is called the *$\mathcal{W}$-(super)algebra associated with the pair $(\mathfrak{g}, f)$ at level k*. By definition, the vertex algebra $\mathcal{W}_k(\mathfrak{g}, f)$ acts naturally on $H^i(V)$, where $V \in \mathcal{O}_k$, $i \in \mathbb{Z}$. Thus, we obtain a family of functors $V \rightsquigarrow H^i(V)$, depending on $i \in \mathbb{Z}$, from $\mathcal{O}_k$ to the category of $\mathcal{W}_k(\mathfrak{g}, f)$-modules.

Remark 5.2.1.

(1) If $\mathfrak{g}$ is a Lie algebra and f is a regular nilpotent element of $\mathfrak{g}$, then $\mathcal{W}_k(\mathfrak{g}, f)$ is identical to $\mathcal{W}_k(\mathfrak{g})$, the $\mathcal{W}$-algebra defined by B. L. Feĭgin and E. V. Frenkel [10].

(2) V. G. Kac, S.-S. Roan and M. Wakimoto gave a more general definition of $\mathcal{W}$-algebras (see Ref. [17] for details).

5.3. As shown in Ref. [17], the vertex algebra $\mathcal{W}_k(\mathfrak{g}, f)$ has a superconformal algebra structure provided that the level k is non-critical, i.e., that $k + h^\vee \neq 0$. Here, $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. Let $L(z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$ be the corresponding Virasoro field. The explicit form of $L(z)$ is given in Ref. [17]. If $f = f_\theta$, its central charge is given by

$$c(k) = \frac{k\,\mathrm{sdim}\,\mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4. \tag{55}$$

Let

$$S(z)=\sum_{n\in\mathbb{Z}}S(n)z^{-n-2}\underset{\text{def}}{=}2(k+h^{\vee})L(z). \tag{56}$$

Then, $S(z)$ is well-defined for any level k.

Remark 5.3.1. Let $\widehat{\Omega}$ be the universal Casimir operator [15] of $\widehat{\mathfrak{g}}$ acting on $V\in\mathcal{O}_k$. Then, we have

$$S(0)+2(k+h^{\vee})\mathbf{D}^{\mathcal{W}}=\widehat{\Omega}$$

on $H^{\bullet}(V)$.

5.4. Let

$$J^{(v)}(z)=\sum_{n\in\mathbb{Z}}J^{(v)}(n)z^{-n-1}=v(z)+\sum_{\beta,\gamma\in\Delta_{>0}}(-1)^{p(\gamma)}([v,u_{\beta}]|u_{-\gamma}):\psi_{\gamma}(z)\psi^{\beta}(z):,$$

for $v\in\mathfrak{g}_{\leq 0}$.

Also, let $C_k(\mathfrak{g})$ be the subspace of $C(V_k(\mathfrak{g}))$ spanned by the vectors

$$J^{(u_1)}(m_1)\dots J^{(u_p)}(m_p)\Phi_{\alpha_1}(n_1)\dots\Phi_{\alpha_q}(n_q)\psi^{\beta_1}(s_1)\dots\dots\psi^{\beta_r}(s_r)|0\rangle$$

with $u_i\in\mathfrak{g}_{\leq 0}$, $\alpha_i\in\Delta_{\frac{1}{2}}$, $\beta_i\in\Delta_{>0}$, $m_i,n_i,s_i\in\mathbb{Z}$. As shown in Ref. [19], $C_k(\mathfrak{g})$ is a vertex subalgebra and a subcomplex of $C(V_k(\mathfrak{g}))$. Moreover, it was proved that

$$\mathcal{W}_k(\mathfrak{g},f)=H^0(C_k(\mathfrak{g}),d) \tag{57}$$

as vertex algebras. This follows from the tensor product decomposition of the complex $C(V_k(\mathfrak{g}))$ (see Ref. [19] for details).

5.5. Let

$$\widehat{\mathfrak{g}}^f\underset{\text{def}}{=}\mathfrak{g}^f\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}1$$

be the affine Lie superalgebra of $\mathfrak{g}^f$ with respect to the 2-cocycle $(\ ,\ )^{\natural}$, defined by

$$(u\otimes t^m,v\otimes t^n)^{\natural}=\begin{cases}m\delta_{m,n}\left((k+h^{\vee})(u|v)-\frac{1}{2}\operatorname{str}_{\mathfrak{g}_0}(\operatorname{ad}u)(\operatorname{ad}v)\right) & (\text{if } u,v\in\mathfrak{g}_0),\\ 0 & (\text{otherwise}).\end{cases} \tag{58}$$

Also, let $V_k^{\natural}(\mathfrak{g}^f)$ be the corresponding universal vertex affine algebra:

$$V_k^{\natural}(\mathfrak{g}^f)\underset{\text{def}}{=}U(\widehat{\mathfrak{g}}^f)\otimes_{U(\mathfrak{g}^f\otimes\mathbb{C}[t]\oplus\mathbb{C}1)}\mathbb{C}. \tag{59}$$

Then, the correspondence

$$v\otimes t^n\to J^{(v)}(n)\quad\text{for } v\otimes t^n\in\widehat{\mathfrak{g}}^f$$

defines a $V_k^{\natural}(\mathfrak{g}^f)$-module structure on $C(V)$, $V\in\mathcal{O}_k$. In particular, we have the following embedding of vertex algebras:

$$V_k^{\natural}(\mathfrak{g}^f)\hookrightarrow C_k(\mathfrak{g})\subset C(V_k(\mathfrak{g})). \tag{60}$$

Theorem 5.5.1 (V. G. Kac and M. Wakimoto: Theorem 4.1 of Ref. [19])**.** *There exists a filtration*

$$\{0\}=F_{-1}\mathcal{W}_k(\mathfrak{g},f)\subset F_0\mathcal{W}_k(\mathfrak{g},f)\subset F_1\mathcal{W}_k(\mathfrak{g},f)\subset\dots$$

of $\mathcal{W}_k(\mathfrak{g},f)=H^0(C_k(\mathfrak{g}),d)$ *such that*

$$\begin{aligned}
&\mathcal{W}_k(\mathfrak{g},f)=\bigcup_p F_p\mathcal{W}_k(\mathfrak{g},f),\\
&\widehat{\mathfrak{t}}\cdot F_p\mathcal{W}_k(\mathfrak{g},f)\subset F_p\mathcal{W}_k(\mathfrak{g},f)\quad \textit{for all } p,\\
&F_p\mathcal{W}_k(\mathfrak{g},f)\cdot F_q\mathcal{W}_k(\mathfrak{g},f)\subset F_{p+q}\mathcal{W}_k(\mathfrak{g},f)\quad \textit{for all } p,\ q,
\end{aligned}\tag{61}$$

where the left-hand-side of (61) *denotes the span of the vectors ,*

$$Y_n(v)w\quad (v\in F_p\mathcal{W}_k(\mathfrak{g},f),\ w\in F_q\mathcal{W}_k(\mathfrak{g},f),n\in\mathbb{Z}),$$

and such that the map (60) *induces the isomorphism*

$$V_k^\natural(\mathfrak{g}^f)\cong \mathrm{gr}_F\,\mathcal{W}_k(\mathfrak{g},f)=\bigoplus F_p\mathcal{W}_k(\mathfrak{g},f)/F_{p-1}\mathcal{W}_k(\mathfrak{g},f)$$

as vertex algebras and $\widehat{\mathfrak{t}}$*-modules.*

Remark 5.5.2. Actually, stronger results were proved by V. G. Kac and M. Wakimoto [19]. Specifically, it was shown that $H^i(V_k(\mathfrak{g}))=H^i(C_k(\mathfrak{g}),d)=\{0\}$ $(i\neq 0)$. Their proof is based on the argument given in Ref. [12]. Further, the explicit form of $\mathcal{W}_k(\mathfrak{g},f)$ was obtained for the case $f=f_\theta$.

For $v\in\mathfrak{g}^f$, let $W^{(v)}\in\mathcal{W}_k(\mathfrak{g},f)$ be the cocycle in $C_k(\mathfrak{g})$ corresponding to $v(-1)|0\rangle\in V_k^\natural(\mathfrak{g}^f)$. Let us write

$$Y(W^{(v)},z)=\sum_{n\in\mathbb{Z}}W^{(v)}(n)z^{-n-1}.\tag{62}$$

Then, by Theorem 5.5.1, the following map defines an isomorphism of $\widehat{\mathfrak{t}}$-modules:

$$\begin{array}{rcl}
U(\mathfrak{g}^f\otimes\mathbb{C}[t^{-1}]t^{-1}) & \to & \mathcal{W}_k(\mathfrak{g},f),\\
u_1(n_1)\dots u_r(n_r) & \mapsto & W^{(u_1)}(n_1)\dots W^{u_r}(n_r)|0\rangle.
\end{array}$$

We remark that the filtration $\{F_p\mathcal{W}_k(\mathfrak{g},f)\}$ in Theorem 5.5.1 is now described by

$$\begin{aligned}
&F_p\mathcal{W}_k(\mathfrak{g},f)\\
&=\mathrm{span}\{W^{(u_1)}(n_1)\dots W^{(u_r)}(n_r)|0\rangle \mid r\geq 0,\ n_i\in\mathbb{Z},\ u_i\in\mathfrak{g}^f_{-s_i},\ \sum_{i=1}^r s_i\leq p\}.
\end{aligned}\tag{63}$$

Remark 5.5.3.

(1) Let $V\in\mathcal{O}_k$. Then, we have

$$W^{(v)}(n)H^\bullet(V)_\xi\subset H^\bullet(V)_{\xi+\eta}\quad (\text{if } v(n)\in(\widehat{\mathfrak{g}}^f)_\eta).$$

Thus, in particular,

$$[\mathbf{D}^{\mathcal{W}},W^{(v)}(n)]=(n-j)W^{(v)}(n)\quad \text{if } v\in\mathfrak{g}^f_{-j}$$

in $\mathrm{End}(H^\bullet(V))$.

(2) $W^{(f)}(n)$ coincides with $S(n-1)$ up to a nonzero multiplicative factor.

6. Irreducible highest weight representations and their characters

We assume that $f=f_\theta$ *and the condition* (32) *is satisfied for the remainder of the paper.*

6.1. Decompose $\widehat{\mathfrak{g}}^f$ as

$$\widehat{\mathfrak{g}}^f = (\widehat{\mathfrak{g}}^f)_- \oplus \widehat{\mathfrak{h}}_{\mathcal{W}} \oplus (\widehat{\mathfrak{g}}^f)_+, \tag{64}$$

where

$$\begin{aligned}
\left(\widehat{\mathfrak{g}}^f\right)_+ &\underset{\text{def}}{=} \mathbb{C}f \otimes \mathbb{C}[t]t^2 \oplus (\mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{n}_{0,-} \oplus \mathfrak{h}^f) \otimes \mathbb{C}[t]t \oplus \mathfrak{n}_{0,+} \otimes \mathbb{C}[t],\\
\widehat{\mathfrak{h}}_{\mathcal{W}} &\underset{\text{def}}{=} \mathfrak{h}^f \oplus \mathbb{C}1 \oplus \mathbb{C}f \otimes \mathbb{C}t,\\
\left(\widehat{\mathfrak{g}}^f\right)_- &\underset{\text{def}}{=} (\mathbb{C}f \oplus \mathfrak{g}_{-\frac{1}{2}} \oplus \mathfrak{n}_{0,-}) \otimes \mathbb{C}[t^{-1}] \oplus (\mathfrak{h}^f \oplus \mathfrak{n}_{0,+}) \otimes \mathbb{C}[t^{-1}]t^{-1}.
\end{aligned}$$

By definition, $\widehat{\mathfrak{h}}_{\mathcal{W}}$ is commutative. Let V be a $\mathcal{W}_k(\mathfrak{g}, f)$-module. Then, the correspondence

$$h \mapsto W^{(h)}(0) \ (h \in \mathfrak{h}^f), \quad f \otimes t^1 \mapsto S(0) \tag{65}$$

defines the action of $\widehat{\mathfrak{h}}_{\mathcal{W}}$ on V (see Ref. [17]). Below we shall regard a $\mathcal{W}_k(\mathfrak{g}, f)$-module as a module over $\widehat{\mathfrak{h}}_{\mathcal{W}}$ via the correspondence (65). Note that, for $k \neq -h^\vee$, the $\widehat{\mathfrak{h}}_{\mathcal{W}}$-action on $H^\bullet(V)$, with $V \in Obj\mathcal{O}_k$, is essentially the same as the $\widehat{\mathfrak{t}}$-action on it.

The following lemma is obvious (see Remark 5.5.3).

Lemma 6.1.1. *The operator $W^{(u)}(n)$, with $u(n) \in (\widehat{\mathfrak{g}}^f)_+$, is locally nilpotent on $H^\bullet(V)$, for $V \in Obj\mathcal{O}_k$.*

6.2. A $\mathcal{W}_k(\mathfrak{g}, f)$-module V is called $\widehat{Q}^{\mathfrak{t}}_+$-*gradable* if it admits a decomposition $V = \bigoplus_{\xi \in \widehat{Q}^{\mathfrak{t}}_+} V[-\xi]$ such that $W^{(v)}(n)V[-\xi] \subset V[-\xi + \eta]$ for $v(n) \in (\widehat{\mathfrak{g}}^f)_\eta$, $\xi \in \widehat{Q}^{\mathfrak{t}}_+$ and $\forall \eta \in \widehat{Q}^{\mathfrak{t}}$. A $\widehat{Q}^{\mathfrak{t}}_+$-gradable $\mathcal{W}_k(\mathfrak{g}, f)$-module V is called a *highest weight module* with *highest weight* $\phi \in \widehat{\mathfrak{h}}^*_{\mathcal{W}}$ if there exists a non-zero vector v_ϕ (called a *highest weight vector*) such that

$$\begin{aligned}
&V \text{ is generated by } v_\phi \text{ over } \mathcal{W}_k(\mathfrak{g}, f),\\
&W^{(u)}(n)v_\phi = 0 \quad (\text{if } u \otimes t^n \in (\widehat{\mathfrak{g}}^f)_+),\\
&W^{(h)}(0)v_\phi = \phi(h)v_\phi \quad (\text{if } h \in \mathfrak{h}^f),\\
&S(0)v_\phi = \phi(f \otimes t)v_\phi.
\end{aligned}$$

Let $B = \{b_j \mid j \in J\}$ be a PBW basis of $U((\widehat{\mathfrak{g}}^f)_-)$ of the form

$$b_j = (u_{j_1} \otimes t^{n_{j_1}}) \dots (u_{j_r} \otimes t^{n_{j_r}}).$$

Then, a highest weight module is spanned by the vectors

$$W^{(b_j)} \underset{\text{def}}{=} W^{(u_{j_1})}(n_{j_1}) \dots W^{(u_{j_r})}(n_{j_r})v_\phi.$$

A highest weight module V with highest weight vector v_ϕ is called a *Verma module* if the above vectors $W^{(b)}$, with $b \in B$, forms a basis of V (see Ref. [19]). Let $\mathbf{M}(\phi)$ denote the Verma module with highest weight $\phi \in \widehat{\mathfrak{h}}^*_{\mathcal{W}}$.

Remark 6.2.1. Let

$$\begin{aligned}
&F_p\mathbf{M}(\phi)\\
&= \operatorname{span}\{W^{(u_1)}(n_1) \dots W^{(u_r)}(n_r)v_\phi \mid r \geq 0,\ n_i \in \mathbb{Z},\ u_i \in \mathfrak{g}^f_{-s_i},\ \sum_{i=1}^r s_i \leq p\}.
\end{aligned}$$

Then, $\{F_p\mathbf{M}(\phi)\}$ defines an increasing filtration of $\mathbf{M}(\phi)$ such that the corresponding graded space $\mathrm{gr}_F\,\mathbf{M}(\phi)$ is naturally a module over $\mathrm{gr}_F\,\mathcal{W}_k(\mathfrak{g},f)$ and isomorphic to $U(\widehat{\mathfrak{g}}^f)\otimes_{U(\widehat{\mathfrak{h}}_{\mathcal{W}}\oplus(\widehat{\mathfrak{g}}^f)_+)}\mathbb{C}_\phi$, where $\mathbb{C}_\phi$ is a $\widehat{\mathfrak{h}}_{\mathcal{W}}\oplus(\widehat{\mathfrak{g}}^f)_+$-module on which $(\widehat{\mathfrak{g}}^f)_+$ acts trivially and $\widehat{\mathfrak{h}}_{\mathcal{W}}$ acts through the character ϕ.

6.3. For $\lambda\in\widehat{\mathfrak{h}}_k^*$, define $\phi_\lambda\in\widehat{\mathfrak{h}}_{\mathcal{W}}^*$ by

$$\begin{aligned}&\phi_\lambda(h)=\lambda(h)\text{ for }h\in\mathfrak{h}^f,\\&\phi_\lambda(f\otimes t)=|\lambda+\rho|^2-|\rho|^2-2(k+h^\vee)\langle\lambda,\mathbf{D}^{\mathcal{W}}\rangle\end{aligned}\tag{66}$$

(cf. Remark 5.3.1). Here, $\rho=\bar\rho+h^\vee\Lambda_0$, and $\bar\rho$ is equal to the half of the difference of the sum of positive even roots and the sum of positive odd roots of $\mathfrak{g}$. Then, the correspondence $\widehat{\mathfrak{h}}_k^*\ni\lambda\mapsto\phi_\lambda\in\widehat{\mathfrak{h}}_{\mathcal{W}}^*$ is surjective.

By Proposition 4.8.1 (1), there is a natural homomorphism of $\mathcal{W}_k(\mathfrak{g},f)$-modules of the form

$$\begin{array}{ccc}\mathbf{M}(\phi_\lambda)&\to&H^0(M(\lambda))\\ v_{\phi_\lambda}&\mapsto&|\lambda\rangle.\end{array}\tag{67}$$

Theorem 6.3.1 (V. G. Kac and M. Wakimoto: Theorem 6.3 of Ref. [19])**.** *For each $\lambda\in\widehat{\mathfrak{h}}_k^*$, with $k\in\mathbb{C}$, we have $H^i(M(\lambda))=\{0\}$ for $i\neq0$, and the map* (67) *is an isomorphism of $\mathcal{W}_k(\mathfrak{g},f)$-modules: $H^0(M(\lambda))\cong\mathbf{M}(\phi_\lambda)$.*

6.4. Let $\phi\in\widehat{\mathfrak{h}}_{\mathcal{W}}^*$. Choose $\lambda\in\widehat{\mathfrak{h}}_k^*$ such that $\phi_\lambda=\phi$. Let $\mathbf{N}(\phi)$ be the the sum of all $\mathbf{D}^{\mathcal{W}}$-stable proper $\mathcal{W}_k(\mathfrak{g},f)$-submodules of $H^0(M(\lambda))=\mathbf{M}(\phi_\lambda)$ (Theorem 6.3.1). Then, $\mathbf{N}(\phi)\subset\mathbf{M}(\phi)$ is independent of the choice of λ, as long as $\phi=\phi_\lambda$ holds. We define

$$\mathbf{L}(\phi)\underset{\mathrm{def}}{=}\mathbf{M}(\phi)/\mathbf{N}(\phi)\quad\text{with }\phi\in\widehat{\mathfrak{h}}_{\mathcal{W}}^*.\tag{68}$$

It is clear that in the case that $k\neq-h^\vee$, $\mathbf{L}(\phi)$ is the unique irreducible quotient of $\mathbf{M}(\phi)$. In the case that $k=-h^\vee$, the following assertion is proved together with Theorem 6.7.4.

Theorem 6.4.1. *The $\mathcal{W}_k(\mathfrak{g},f)$-module $\mathbf{L}(\phi)$, with $\phi\in\widehat{\mathfrak{h}}_{\mathcal{W}}^*$, is irreducible.*

It is clear that the set $\{\mathbf{L}(\phi)\mid\phi\in\widehat{\mathfrak{h}}_{\mathcal{W}}^*\}$ represents the complete set of the isomorphism classes of irreducible highest weight modules over $\mathcal{W}_k(\mathfrak{g},f)$.

6.5.

Theorem 6.5.1. *For any object V in $Obj\mathcal{O}_k^{\triangle}$, we have $H^i(V)=\{0\}$ $(i\neq0)$. In particular, we have $H^i(P_{\leq\lambda}(\mu))=\{0\}$ $(i\neq0)$ for any $\lambda,\mu\in\widehat{\mathfrak{h}}_k^*$ such that $\mu\leq\lambda$.*

Proof. By Theorem 6.3.1, the assertion can be shown by induction applied to the length of a Verma flag of $V\in Obj\mathcal{O}_k^{\triangle}$ (cf. Theorem 8.1 of Ref. [1]). □

Theorem 6.5.2. *For any object V of $\mathcal{O}_k$ we have $H^i(V)=\{0\}$ for all $i>0$.*

Proof. We may assume that $V\in Obj\mathcal{O}_k^{\leq\lambda}$ for some $\lambda\in\widehat{\mathfrak{h}}_k^*$. Also, by Proposition 4.4.2, we may assume that $V\in Obj\dot{\mathcal{O}}_k^{\leq\lambda}$, since the cohomology functor commutes with injective limits. Clearly, it is sufficient to show that $H_i(V)_\xi=\{0\}$ $(i>0)$ for each $\xi\in\widehat{\mathfrak{t}}^*$. By Lemma 4.5.2, for a given ξ, there exists a finitely generated submodule V' of V such that

$$H^i(V)_\xi=H^i(V')_\xi\quad\text{for all }i\in\mathbb{Z}.\tag{69}$$

Because V' is finitely generated, there exists a projective module P of the form $\bigoplus_{i=1}^{r} P_{\leq\lambda}(\mu_i)$ and an exact sequence $0 \to N \to P \to V \to 0$ in $\dot{\mathcal{O}}_k^{\leq\lambda}$. Considering the corresponding long exact sequence, we obtain

$$\cdots \to H^i(P) \to H^i(V) \to H^{i+1}(N) \to H^{i+1}(P) \to \dots . \tag{70}$$

Hence, it follows that $H^i(V') \cong H^{i+1}(N)$ for all $i > 0$, by Proposition 6.5.1. Therefore, we find

$$H^i(V)_\xi \cong H^{i+1}(N)_\xi \quad \text{for all } i > 0, \tag{71}$$

by (69). Then, because $N \in Obj\dot{\mathcal{O}}_k^{\leq\lambda}$, we can repeat this argument to find, for each $k > 0$, some object N_k of $\dot{\mathcal{O}}_k^{\leq\lambda}$ such that

$$H^i(V)_\xi \cong H^{i+k}(N_k)_\xi \quad \text{for all } i > 0. \tag{72}$$

But by Proposition 4.10.1, this implies that $H^i(V)_\xi = \{0\}$ for $i > 0$. □

6.6.

Theorem 6.6.1. *For any $\lambda \in \widehat{\mathfrak{h}}_k^*$, we have $H^i(M(\lambda)^*) = \{0\}$ for $i \neq 0$.*

Theorem 6.6.1 is proved in Subsection 7.17.

Theorem 6.6.2. *Suppose that $\langle \lambda, \alpha_0^\vee \rangle \notin \{0, 1, 2, \dots\}$. Then, any nonzero $\mathcal{W}_k(\mathfrak{g}, f)$-submodule of $H^0(M(\lambda)^*)$ contains the canonical vector $|\lambda\rangle^*$ of $H^0(M(\lambda)^*)$.*

Theorem 6.6.2 is proved together with Theorem 7.16.1.

Remark 6.6.3. Theorem 6.6.2 holds without any restriction on λ. Indeed, it can seen from Corollary 6.7.3 and (the proof of) Theorem 6.7.4 that if $\langle \lambda, \alpha_0^\vee \rangle \in \{0, 1, 2, \dots\}$, then $H^0(M(\lambda)^*) \cong H^0(M(r_0 \circ \lambda)^*)$, where r_0 is the reflection corresponding to α_0.

The following theorem is a consequence of Theorem 6.6.1 which can be proved in the same manner as Theorem 8.1 of Ref. [1].

Theorem 6.6.4. *For a given $\lambda \in \widehat{\mathfrak{h}}_k^*$, $H^i(I_{\leq\lambda}(\mu)) = \{0\}$ $(i \neq 0)$ for any $\lambda, \mu \in \widehat{\mathfrak{h}}_k^*$ such that $\mu \leq \lambda$.*

Using Theorem 6.6.4, the following assertion can be proved in the same manner as Theorem 6.5.1.

Theorem 6.6.5. *For any object V of $\mathcal{O}_k$ we have $H^i(V) = \{0\}$ for all $i < 0$.*

6.7. **Main results.** From Theorems 6.5.2 and 6.6.5, we obtain the following results.

Theorem 6.7.1. *Let k be an arbitrary complex number. We have $H^i(V) = \{0\}$ $(i \neq 0)$ for any object V in $\mathcal{O}_k$.*

Remark 6.7.2. By Theorem 6.7.1, we have, in particular, $H^i(L(\lambda)) = \{0\}$ $(i \neq 0)$ for each $\lambda \in \widehat{\mathfrak{h}}^*$. This was conjectured by V. G. Kac, S.-S. Roan and M. Wakimoto [17] in the case that λ is admissible.

Corollary 6.7.3. *For any $k \in \mathbb{C}$, the correspondence $V \rightsquigarrow H^0(V)$ defines an exact functor from $\mathcal{O}_k$ to the category of $\mathcal{W}_k(\mathfrak{g}, f)$-modules.*

Theorem 6.7.4. *Let k be an arbitrary complex number and let $\lambda \in \widehat{\mathfrak{h}}_k^*$. If $\langle \lambda, \alpha_0^\vee \rangle \in \{0, 1, 2, \dots\}$, then $H^0(L(\lambda)) = \{0\}$. Otherwise $H^0(L(\lambda))$ is an irreducible $\mathcal{W}_k(\mathfrak{g}, f)$-module that is isomorphic to $\mathbf{L}(\phi_\lambda)$.*

Proof. Let $N(\lambda)$ be the unique maximal proper submodule of $M(\lambda)$. Then, we have an exact sequence $0 \to N(\lambda) \to M(\lambda) \to L(\lambda) \to 0$ in $\mathcal{O}_k$. With this, using Corollary 6.7.3, we obtain an exact sequence

$$0 \to H^0(N(\lambda)) \to H^0(M(\lambda)) \xrightarrow{\pi} H^0(L(\lambda)) \to 0. \tag{73}$$

By Theorem 6.3.1, we have $H^0(M(\lambda)) \cong \mathbf{M}(\phi_\lambda)$. Therefore (73) shows that $H^0(L(\lambda))$ is generated over $\mathcal{W}_k(\mathfrak{g}, f)$ by the image $\pi(|\lambda\rangle)$ of the highest weight vector $|\lambda\rangle \in H^0(M(\lambda))$. Thus it follows that $H^0(L(\lambda)) = \{0\}$ if and only if $\pi(|\lambda\rangle) \neq \{0\}$. But $\pi(|\lambda\rangle) \neq \{0\}$ if and only if $H^0(L(\lambda))_{\xi_\lambda} \neq \{0\}$, because $\dim_{\mathbb{C}} H^0(M(\lambda))_{\xi_\lambda} = 1$, by Proposition 4.8.1 (1). Hence, by Proposition 4.8.1 (2), it follows that $H^0(L(\lambda))$ is nonzero if and only if $\langle \lambda, \alpha_0^\vee \rangle \notin \{0, 1, 2, \dots\}$.

Next, suppose that $\langle \lambda, \alpha_0^\vee \rangle \notin \{0, 1, 2, \dots\}$, so that $H^0(L(\lambda)) \neq \{0\}$. Also, let N be a nonzero submodule of $H^0(L(\lambda))$. As $L(\lambda)$ is a submodule of $M(\lambda)^*$, $H^0(L(\lambda))$ is also a submodule of $H^0(M(\lambda)^*)$, by the exactness of the functor $H^0(?)$ (Corollary 6.7.3). Hence, N is a submodule of $H^0(M(\lambda)^*)$. But then, by Theorem 6.6.2, it follows that $\pi(|\lambda\rangle) \in N$. Therefore N must coincide with the entire space $H^0(L(\lambda))$. We have thus shown that $H^0(L(\lambda))$ is irreducible. Finally, we have to show (in the case $k = -h^\vee$) that $\mathbf{N}(\phi_\lambda) = H^0(N(\lambda))$ ($\mathbf{N}(\phi)$ is defined in Section 6.4.) Clearly, $H^0(N(\lambda))$ is stable under the action of $\mathbf{D}^{\mathcal{W}}$. Thus, $\mathbf{N}(\phi_\lambda) \supset H^0(N(\lambda))$. In addition, $\mathbf{N}(\phi_\lambda) \subset H^0(N(\lambda))$, by the irreduciblity of $H^0(L(\lambda)) = H^0(M(\lambda))/H^0(N(\lambda))$, which we have just proved. This completes the proof. □

Remark 6.7.5. Theorem 6.7.4 was conjectured by V. G. Kac, S.-S. Roan, M. Wakimoto [17] in the case of an admissible weight λ.

6.8. **The characters.** For an object V of $\dot{\mathcal{O}}_k$, define the formal characters

$$\operatorname{ch} V \underset{\text{def}}{=} \sum_{\lambda \in \widehat{\mathfrak{h}}^*} e^\lambda \dim_{\mathbb{C}} V^\lambda,$$

$$\operatorname{ch} H^0(V) \underset{\text{def}}{=} \sum_{\xi \in \widehat{\mathfrak{t}}^*} e^\xi \dim_{\mathbb{C}} H^0(V)_\xi.$$

In addition, for $\lambda, \mu \in \widehat{\mathfrak{h}}^*$ define an integer $[L(\lambda) : M(\mu)]$ by

$$\operatorname{ch} L(\lambda) = \sum_{\mu \in \widehat{\mathfrak{h}}^*} [L(\lambda) : M(\mu)] \operatorname{ch} M(\mu).$$

Then, by the exactness of the functor $H^0(?)$ (see Corollary 6.7.3), we have

$$\operatorname{ch} H^0(L(\lambda)) = \sum_{\mu \in \widehat{\mathfrak{h}}^*} [L(\lambda) : M(\mu)] \operatorname{ch} H^0(M(\mu)). \tag{74}$$

Because the correspondence $\widehat{\mathfrak{h}}_k^* \ni \lambda \mapsto \phi_\lambda \in \widehat{\mathfrak{h}}_{\mathcal{W}}^*$ (see (66)) is surjective, it follows from (74), Theorem 6.3.1 and Theorem 6.7.4 that the character of *any* irreducible highest weight representation $\mathbf{L}(\phi)$ of $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ at *any* level $k \in \mathbb{C}$ is determined by the character of the corresponding $\widehat{\mathfrak{g}}$-module $L(\lambda)$.

Remark 6.8.1.

(1) For an admissible weight λ, the character formula (74) was conjectured by V. G. Kac, S.-S. Roan, M. Wakimoto [17].

(2) By Theorem 6.7.1, it follows that (74) can be obtained by calculating the Euler-Poincaré character of $H^0(L(\lambda))$ (see Ref. [17] for details).
(3) In the case that $\mathfrak{g}$ is a Lie algebra and $k \neq -h^\vee$, the number $[L(\lambda) : M(\mu)]$ is known. (It can be expressed in terms of the Kazhdan-Lusztig polynomials. The most general formula is given in Ref. [21]).
(4) In the case that $\mathfrak{g} = \mathfrak{spo}(2|1)$, $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is the Neveu-Schwarz algebra. All minimal series representations of the Neveu-Schwarz algebra (see, e.g., Ref. [18]) can be obtained from the admissible $\widehat{\mathfrak{spo}(2|1)}$-modules [18], as explained by V. G. Kac, S.-S. Roan and M. Wakimoto [17].
(5) In the case that $\mathfrak{g} = \mathfrak{sl}(2|1)$, $\mathcal{W}_k(\mathfrak{g}, f_\theta)$ is the $N = 2$ superconformal algebra. The minimal series representations of the $N = 2$ superconformal algebra (cf. Refs. [6, 23, 26]) can be obtained from the admissible $\widehat{\mathfrak{sl}(2|1)}$-modules [18], as explained by V. G. Kac, S.-S. Roan and M. Wakimoto [17].
(6) For further examples and references, see Refs. [17, 19, 20].

7. The computation of $H^\bullet(M(\lambda)^*)$

In this section, we prove Theorems 6.6.1 and 6.6.2. Specifically, we compute $H^\bullet(M(\lambda)^*)$, with $\lambda \in \widehat{\mathfrak{h}}_k^*$. This is done by using a spectral sequence which we define in Subsection 7.11. It is a version of the Hochschild-Serre spectral sequence for the subalgebra $\mathbb{C}e(-1) \oplus \mathfrak{g}_{>0} \otimes \mathbb{C}[t] \subset L\mathfrak{g}_{>0}$.

7.1. As in (14), let

$$N(\chi_-) \underset{\text{def}}{=} U(L\mathfrak{g}_{<0})/U(L\mathfrak{g}_{<0}) \ker \chi_-. \tag{75}$$

Here, $\ker \chi_- \subset U(L\mathfrak{g}_{\leq -1})$ is the kernel of the character χ_- of $L\mathfrak{g}_{\leq -1}$, defined by

$$\chi_-(u(m)) \underset{\text{def}}{=} (e(-1)|u(m)), \text{ where } u \in \mathfrak{g}_{\leq -1} \text{ and } m \in \mathbb{Z}. \tag{76}$$

Let $\Phi_u(n)$, with $u \in \mathfrak{g}_{<0}$ and $n \in \mathbb{Z}$, denote the image of $u(n) \in L\mathfrak{g}_{<0}$ in $N(\chi_-)$. As above, we set $\Phi_{-\alpha}(n) = \Phi_{u_{-\alpha}}(n)$ with $\alpha \in \Delta_{\frac{1}{2}}$ and $n \in \mathbb{Z}$. Then, the correspondence $\Phi_\alpha(n) \mapsto \Phi_{-\alpha}(-n)$, with $\alpha \in \Delta_{\frac{1}{2}}$ and $n \in \mathbb{Z}$, defines the anti-algebra isomorphism

$$N(\chi) \cong N(\chi_-).$$

7.2. Let $\mathcal{F}^{\text{ne}}(\chi_-)$ be the irreducible representation of $N(\chi_-)$ generated by a vector $\mathbf{1}_{\chi_-}$ such that $\Phi_{-\alpha}(n)\mathbf{1}_{\chi_-} = 0$ for $\alpha \in \Delta_{\frac{1}{2}}$ and $n \geq 1$. As in the case considered above, we define a semisimple action of $\widehat{\mathfrak{h}}$ on $\mathcal{F}^{\text{ne}}(\chi_-)$ by $h\mathbf{1}_{\chi_-} = 0$, $\Phi_{-\alpha}(n)\mathcal{F}^{\text{ne}}(\chi_-)^\lambda \subset \mathcal{F}^{\text{ne}}(\chi_-)^{\lambda-\alpha+n\delta}$, with $h \in \mathfrak{h}$, $\alpha \in \Delta_{\frac{1}{2}}$, $n \leq 0$, and $\lambda \in \widehat{\mathfrak{h}}^*$. Then, $\mathcal{F}^{\text{ne}}(\chi_-) = \bigoplus_{\xi \in \widehat{\mathfrak{t}}^*} \mathcal{F}^{\text{ne}}(\chi_-)_\xi$ and $\dim \mathcal{F}^{\text{ne}}(\chi_-)_\xi < \infty$ for all ξ.

7.3. There exists a unique bilinear form

$$\langle \cdot | \cdot \rangle^{\text{ne}} : \mathcal{F}^{\text{ne}}(\chi) \times \mathcal{F}^{\text{ne}}(\chi_-) \to \mathbb{C} \tag{77}$$

such that $\langle \mathbf{1}_\chi | \mathbf{1}_{\chi_-} \rangle^{\text{ne}} = 1$ and $\langle \Phi_\alpha(m)v|v' \rangle^{\text{ne}} = \langle v|\Phi_{-\alpha}(-m)v' \rangle^{\text{ne}}$, where $v \in \mathcal{F}^{\text{ne}}(\chi)$, $v' \in \mathcal{F}^{\text{ne}}(\chi_-)$, $\alpha \in \Delta_{\frac{1}{2}}$ and $m \in \mathbb{Z}$. It is easy to see that this form is non-degenerate. Indeed, its restriction on $\mathcal{F}^{\text{ne}}(\chi)_\xi \times \mathcal{F}^{\text{ne}}(\chi_-)_\xi$, with $\xi \in \widehat{\mathfrak{t}}^*$, is non-degenerate. Hence, we have

$$\mathcal{F}^{\text{ne}}(\chi) = \mathcal{F}^{\text{ne}}(\chi_-)^*, \tag{78}$$

since each space $\mathcal{F}^{\mathrm{ne}}(\chi_-)_\xi$, with $\xi \in \widehat{\mathfrak{t}}^*$, decomposes into a finite sum of finite-dimensional weight spaces $\mathcal{F}^{\mathrm{ne}}(\chi_-)^\lambda$.

7.4. Let $\mathcal{C}l(L\mathfrak{g}_{<0})$ be the Clifford superalgebra associated with $L\mathfrak{g}_{<0} \oplus (L\mathfrak{g}_{<0})^*$ and its natural bilinear form. It is generated by the elements $\psi_{-\alpha}(n)$ and $\psi^{-\alpha}(n)$ with $\alpha \in \Delta_{<0}$ and $n \in \mathbb{Z}$ which satisfy the relations $[\psi_{-\alpha}(m), \psi^{-\beta}(n)] = \delta_{\alpha,\beta}\delta_{m+n,0}$. Here the parity of $\psi_{-\alpha}(n)$ and $\psi^{-\alpha}(n)$ is reverse to $u_{-\alpha}$. We have an anti-algebra isomorphism $\mathcal{C}l(L\mathfrak{g}_{>0}) \cong \mathcal{C}l(L\mathfrak{g}_{<0})$ defined by $\psi_\alpha(m) \mapsto (-1)^{p(\alpha)}\psi_{-\alpha}(-m)$ and $\psi^\alpha(m) \mapsto \psi^{-\alpha}(-m)$, where $\alpha \in \Delta_{>0}$ and $m \in \mathbb{Z}$.

7.5. Let $\mathcal{F}(L\mathfrak{g}_{<0})$ be the irreducible representation of $\mathcal{C}l(L\mathfrak{g}_{<0})$ generated by the vector $\mathbf{1}_-$ with the properties $\psi_{-\alpha}(n)\mathbf{1}_- = 0$, where $\alpha \in \Delta_{>0}$, $n \geq 1$, and $\psi^{-\alpha}(n)\mathbf{1}_- = 0$, where $\alpha \in \Delta_{>0}$, $n \geq 0$. As above, we have a natural action of $\widehat{\mathfrak{h}}$ on $\mathcal{F}(L\mathfrak{g}_{<0})$.

There exists a unique bilinear form

$$\langle \cdot | \cdot \rangle^{\mathrm{ch}} : \mathcal{F}(L\mathfrak{g}_{>0}) \times \mathcal{F}(L\mathfrak{g}_{<0}) \to \mathbb{C}, \tag{79}$$

which is non-degenerate on $\mathcal{F}(L\mathfrak{g}_{>0})^\lambda \times \mathcal{F}(L\mathfrak{g}_{<0})^\lambda$, with $\lambda \in \widehat{\mathfrak{h}}^*$, such that $\langle \mathbf{1}|\mathbf{1}_-\rangle^{\mathrm{ch}} = 1$, $\langle \psi_\alpha(n)v|v'\rangle^{\mathrm{ch}} = (-1)^{p(\alpha)}\langle v|\psi_{-\alpha}(-n)v'\rangle^{\mathrm{ch}}$ and $\langle \psi^\alpha(n)v|v'\rangle^{\mathrm{ch}} = \langle v|\psi^{-\alpha}(-n)v'\rangle^{\mathrm{ch}}$, where $v \in \mathcal{F}(L\mathfrak{g}_{>0})$, $v' \in \mathcal{F}(L\mathfrak{g}_{<0})$, $\alpha \in \Delta_{>0}$ and $n \in \mathbb{Z}$. Hence, we have

$$\mathcal{F}(L\mathfrak{g}_{>0}) = \mathcal{F}(L\mathfrak{g}_{<0})^*. \tag{80}$$

7.6. Let

$$C_-(V) = V \otimes \mathcal{F}^{\mathrm{ne}}(\chi_-) \otimes \mathcal{F}(L\mathfrak{g}_{<0}) \quad \text{with } V \in Obj\mathcal{O}_k.$$

Then, $C_-(V) = \bigoplus_{\lambda \in \widehat{\mathfrak{h}}^*} C_-(V)^\lambda$ with respect to the diagonal action of $\widehat{\mathfrak{h}}$. By (78) and (79), we have

$$C(V^*) = C_-(V)^* \quad \text{for } V \in Obj\mathcal{O}_k \tag{81}$$

as $\mathbb{C}$-vector spaces. Here again, * is defined by (11). Under the identification (81), we have

$$(dg)(v) = g(d_- v) \quad (g \in C(V^*), v \in C(V)), \tag{82}$$

where

$$\begin{aligned} d_- = &\sum_{\substack{\alpha \in \Delta_{>0} \\ n \in \mathbb{Z}}} (-1)^{p(\alpha)}(u_{-\alpha}(-n) + \Phi_{u_{-\alpha}}(-n))\psi^{-\alpha}(n) \\ &- \frac{1}{2} \sum_{\substack{\alpha,\beta.\gamma \in \Delta_{>0} \\ k+l+m=0}} (-1)^{p(\alpha)p(\gamma)}(u_\gamma|[u_{-\alpha}, u_{-\beta}])\psi^{-\alpha}(k)\psi^{-\beta}(l)\psi_{-\gamma}(m). \end{aligned}$$

Clearly, we have $d_-^2 = 0$. Also, d_- decomposes as

$$\begin{aligned} &d_- = d_-^\chi + d_-^{\mathrm{st}}, \\ &(d_-^\chi)^2 = (d_-^{\mathrm{st}})^2 = \{d_-^\chi, d_-^{\mathrm{st}}\} = 0, \end{aligned} \tag{83}$$

where

$$d_-^\chi = \sum_{\substack{\alpha \in \Delta_{\frac{1}{2}} \\ n \geq 1}} (-1)^{p(\alpha)}\Phi_{-\alpha}(n)\psi^{-\alpha}(-n) + \sum_{\alpha \in \Delta_1} (-1)^{p(\alpha)}\chi_-(u_{-\alpha}(1))\psi^{-\alpha}(-1) \tag{84}$$

and $d_-^{\mathrm{st}} = d_- - d_-^\chi$.

Remark 7.6.1. By Theorem 2.3 of [13], the complex $(C_-(V), d_-)$ is acyclic for any $V \in Obj\mathcal{O}_k$, since $f(1)$ is locally nilpotent on V (see Remark 7.12.2).

7.7. It is clear that $C_-(V_k(\mathfrak{g}))$ possesses a natural vertex algebra structure. The correspondences $v(n) \mapsto v^t(-n)$, $\psi_\alpha(n) \mapsto (-1)^{p(\alpha)}\psi_{-\alpha}(-n)$, $\psi^\alpha(n) \mapsto \psi^{-\alpha}(-n)$, $\Phi_\alpha(n) \mapsto \Phi_{-\alpha}(-n)$ extend to the anti-algebra homomorphism

$$ {}^t: \ \mathcal{U}(C(V_k(\mathfrak{g})) \to \mathcal{U}(C_-(V_k(\mathfrak{g}))), \tag{85}$$

where $\mathcal{U}(C(V_k(\mathfrak{g})))$ and $\mathcal{U}(C_-(V_k(\mathfrak{g})))$ are universal enveloping algebras of $C(V_k(\mathfrak{g}))$ and $C_-(V_k(\mathfrak{g}))$ respectively in the sense of Ref. [14]. Note that we have the relations $d_- = d^t$, $d_-^{\mathrm{st}} = (d^{\mathrm{st}})^t$ and $d_-^\chi = (d^\chi)^t$.

7.8. For $v \in \mathfrak{g}$, define

$$\begin{aligned} J_-^{(v)}(z) &= \sum_{n\in\mathbb{Z}} J_-^{(v)}(n) z^{-n-1} \\ &\underset{\mathrm{def}}{=} v(z) + \sum_{\alpha,\beta\in\Delta_{>0}} (-1)^{p(\gamma)}(u_\gamma|[v,u_{-\beta}]) : \psi_{-\gamma}(z)\psi^{-\beta}(z) :, \end{aligned}$$

where $\psi_{-\alpha}(z) = \sum_{n\in\mathbb{Z}} \psi_{-\alpha}(n) z^{-n}$ and $\psi^{-\alpha}(z) = \sum_{n\in\mathbb{Z}} \psi^{-\alpha}(z) z^{-n-1}$ with $\alpha \in \Delta_{>0}$. Then, we have

$$J^{(v)}(n)^t = J_-^{(v^t)}(-n) \quad \text{for } v \in \mathfrak{g}^f,\ n \in \mathbb{Z}. \tag{86}$$

7.9. Let $\lambda \in \widehat{\mathfrak{h}}_k^*$ and let $C_-(\lambda)$ be the subspace of $C_-(M(\lambda))$ spanned by the vectors

$$J_-^{(u_1)}(m_1)\dots J_-^{(u_p)}(m_p)\Phi_{-\alpha_1}(n_1)\dots\Phi_{-\alpha_q}(n_q)\psi^{-\beta_1}(s_1)\dots\dots\psi^{-\beta_r}(s_r)|\lambda\rangle_-,$$

with $u_i \in \mathfrak{g}_{\geq 0}$, $\alpha_i \in \Delta_{\frac{1}{2}}$, $\beta_i \in \Delta_{>0}$ and $m_i, n_i, s_i \in \mathbb{Z}$, where $|\lambda\rangle_-$ is the canonical vector $v_\lambda \otimes \mathbf{1}_{\chi_-} \otimes \mathbf{1}_-$. Then, $d_- C_-(\lambda) \subset C_-(\lambda)$, i.e., $C_-(\lambda)$ is a subcomplex of $C_-(M(\lambda))$.

The inclusion $C_-(\lambda) \hookrightarrow C_-(M(\lambda))$ induces the surjection

$$C_-(M(\lambda))^* \twoheadrightarrow C_-(\lambda)^*. \tag{87}$$

Let the differential d act on $C_-(\lambda)^*$ as $(dg)(v) = g(d_- v)$ with $g \in C_-(\lambda)^*$, $v \in C_-(\lambda)$. Then, the space $H^i(C_-(\lambda)^*, d)$, where $i \in \mathbb{Z}$, is naturally a module over $\mathcal{W}_k(\mathfrak{g}, f) = H^0(C_k(\mathfrak{g}))$, and (87) induces the homomorphism

$$H^\bullet(M(\lambda)^*) \to H^\bullet(C_-(\lambda)^*, d) \tag{88}$$

of $\mathcal{W}_k(\mathfrak{g}, f)$-modules. The action of $\mathcal{W}_k(\mathfrak{g}, f)$ on $H^i(C_-(\lambda)^*, d)$ is described by

$$(W^{(v)}(n)g)(v) = g(W^{(v^t)}(-n)v), \tag{89}$$

where $W_-^{(v^t)}(-n)$ is the image of $W^{(v)}(n) \in H^0(\mathcal{U}(C_k(\mathfrak{g})), \operatorname{ad} d)$ under the map (85).

The following proposition can be shown in the same manner as Proposition 6.3 of [1].

Proposition 7.9.1. *For any $\lambda \in \widehat{\mathfrak{h}}^*$, the map* (88) *is an isomorphism of $\mathcal{W}_k(\mathfrak{g}, f)$-modules and $\widehat{\mathfrak{t}}$-modules:*

$$H^\bullet(M(\lambda)^*) \cong H^\bullet(C_-(\lambda)^*, d).$$

Below we compute the cohomology $H^\bullet(C_-(\lambda)^*) = H^\bullet(C_-(\lambda)^*, d)$.

7.10. Here we employ the notation of the previous subsections. First, note that

$$C_-(\lambda) = \bigoplus_{\xi \leq \xi_\lambda} C_-(\lambda)_\xi.$$

Also, observe that the subcomplex $C_-(\lambda)_{\xi_\lambda} \subset C_-(\lambda)$ is spanned by the vectors

$$J_-^{(e_\theta)}(-1)^n|\lambda\rangle_-, \quad J_-^{(e_\theta)}(-1)^n\psi^{-\theta}(-1)|\lambda\rangle_-$$

with $n \in \mathbb{Z}_{\geq 0}$. (Hence, $C_-(\lambda)_{\xi_\lambda}$ is infinite dimensional.) Let

$$G_pC_-(\lambda)_{\xi_\lambda} = \sum_{\substack{\mu\in\widehat{\mathfrak{h}}^* \\ -\langle \mu-\lambda, x\rangle \leq p}} C_-(\lambda)^\mu_{\xi_\lambda} \subset C_-(\lambda)_{\xi_\lambda} \quad \text{for } p \leq 0. \tag{90}$$

Thus, $G_pC_-(\lambda)_{\xi_\lambda}$ is a subspace of $C_-(\lambda)_{\xi_\lambda}$ spanned by the vectors

$$J_-^{(e_\theta)}(-1)^n|\lambda\rangle_-, \quad J_-^{(e_\theta)}(-1)^{n-1}\psi^{-\theta}(-1)|\lambda\rangle_-, \quad \text{with } n \geq -p.$$

Now, the following assertion is clear.

Lemma 7.10.1. *The space $C_-(\lambda)_{\xi_\lambda}/G_pC_-(\lambda)_{\xi_\lambda}$ is finite dimensional for each $p \leq 0$.*

Define $G_pC_-(\lambda)$, with $p \leq 0$, as the subspace of $C_-(\lambda)$ spanned by the vectors

$$J_-^{(u_1)}(m_1)\dots J_-^{(u_p)}(m_p)\Phi_{-\alpha_1}(n_1)\dots\Phi_{-\alpha_q}(n_q)\psi^{-\beta_1}(s_1)\dots\dots\psi^{-\beta_r}(s_r)v,$$

with $u_i \in \mathfrak{g}_{\geq 0}$, $\alpha_i \in \Delta_{\frac{1}{2}}$, $\beta_i \in \Delta_{>0}$, $m_i, n_i, s_i \in \mathbb{Z}$ and $v \in G_pC_-(\lambda)_{\xi_\lambda}$. Then, we have

$$\begin{gathered} \dots \subset G_pC_-(\lambda) \subset G_{p+1}C_-(\lambda) \subset \dots \subset G_0C_-(\lambda) = C_-(\lambda), \\ \bigcap_p G_pC_-(\lambda) = \{0\}, \\ d_-G_pC_-(\lambda) \subset G_pC_-(\lambda). \end{gathered} \tag{91}$$

The following Lemma is easily proven.

Lemma 7.10.2.

(1) *The subspace $G_pC_-(\lambda)$, with $p \leq 0$, is preserved under the the action of the operators $J_-^{(u)}(n)$ with $u \in \mathfrak{g}_{\geq 0}$, $n \in \mathbb{Z}$, $\Phi_{-\alpha}(n)$ with $\alpha \in \Delta_{\frac{1}{2}}$, $n \in \mathbb{Z}$ and $\psi^{-\alpha}(n)$ with $\alpha \in \Delta_{>0}$, $n \in \mathbb{Z}$.*

(2) *For each $p \leq 0$, $C_-(\lambda)/G_pC_-(\lambda)$ is a direct sum of finite dimensional weight spaces $(C_-(\lambda)/G_pC_-(\lambda))_\xi$, with $\xi \in \widehat{\mathfrak{t}}^*$.*

7.11. For a semisimple $\widehat{\mathfrak{t}}$-module X, we define

$$D(X) \underset{\text{def}}{=} \bigoplus_\xi \mathrm{Hom}_{\mathbb{C}}(X_\xi, \mathbb{C}). \tag{92}$$

Next, let

$$G^pC_-(\lambda)^* \underset{\text{def}}{=} (C_-(\lambda)/G_pC_-(\lambda))^* \subset C_-(\lambda)^* \quad \text{for } p \leq 0, \tag{93}$$

where * is defined by (11). Then, by Lemma 7.10.2, we have

$$G^pC_-(\lambda)^* = D(C_-(\lambda)/G_pC_-(\lambda)),$$

and $G^pC_-(\lambda)^*$ is a $C_k(\mathfrak{g})$-submodule of $C_-(\lambda)^*$. Also, by (91), we have

$$\begin{aligned}
&\cdots \supset G^pC_-(\lambda)^* \supset G^{p+1}C_-(\lambda)^* \supset \cdots \supset G^0C_-(\lambda)^* = \{0\},\\
&C_-(\lambda)^* = \bigcup_p G^pC_-(\lambda)^*, \qquad (94)\\
&dG^pC_-(\lambda)^* \subset G^pC_-(\lambda)^*.
\end{aligned}$$

Therefore we obtain the corresponding spectral sequence $E_r \Rightarrow H^\bullet(C_-(\lambda)^*) = H^\bullet(M(\lambda)^*)$. By definition,

$$E_1^{\bullet,q} = H^q(\operatorname{gr}^G C_-(\lambda)^*, d), \tag{95}$$

where $\operatorname{gr}^G C_-(\lambda)^* = \sum_p G^pC_-(\lambda)^*/G^{p+1}C_-(\lambda)^*$. Moreover, because our filtration is compatible with the action of $\widehat{\mathfrak{t}}$, each E_r is a direct sum of $\mathfrak{t}$-weight spaces:

$$E_r = \bigoplus_{\xi \in \widehat{\mathfrak{t}}^*} (E_r)_\xi.$$

In particular, we have

$$(E_r)_\xi \Rightarrow H^\bullet(C_-(\lambda)^*)_\xi = H^\bullet(M(\lambda)^*)_\xi \quad \text{for each } \xi \in \widehat{\mathfrak{t}}^*. \tag{96}$$

Below we compute this spectral sequence.

7.12. Consider the exact sequence

$$0 \to G_{p+1}C_-(\lambda)/G_pC_-(\lambda) \to C_-(\lambda)/G_pC_-(\lambda) \to C_-(\lambda)/G_{p+1}C_-(\lambda) \to 0,$$

where $p \geq -1$. This induces the exact sequence

$$0 \to G^{p+1}C_-(\lambda)^* \to G^pC_-(\lambda)^* \to (G_{p+1}C_-(\lambda)/G_pC_-(\lambda))^* \to 0$$

Therefore

$$\begin{aligned}
G^pC_-(\lambda)^*/G^{p+1}C_-(\lambda)^* &= (G_{p+1}C_-(\lambda)/G_pC_-(\lambda))^*\\
&= D\left(G_{p+1}C_-(\lambda)/G_pC_-(\lambda)\right).
\end{aligned} \tag{97}$$

Here, the last equality follows from Lemma 7.10.2 (2). Again by Lemma 7.10.2 (2), we have the following proposition.

Proposition 7.12.1. *We have*

$$H^i(G^pC_-(\lambda)^*/G^{p+1}C_-(\lambda)^*) = D(H^{-i}(G_{p+1}C_-(\lambda)/G_pC_-(\lambda))$$

for each i and p.

Remark 7.12.2. It is not the case that $H^i(C_-(\lambda)^*) = D(H^{-i}(C_-(\lambda)))$.

7.13. Consider the subcomplex

$$\operatorname{gr}^G C_-(\lambda)_{\xi_\lambda} \subset \operatorname{gr}^G C_-(\lambda) \underset{\text{def}}{=} \sum_p G_pC_-(\lambda)/G_{p-1}C_-(\lambda).$$

By definition, we have

$$\operatorname{gr}^G C_-(\lambda)_{\xi_\lambda} = \bigoplus_p G_pC_-(\lambda)_{\xi_\lambda}/G_{p-1}C_-(\lambda)_{\xi_\lambda},$$

and d_-^χ acts trivially on $\operatorname{gr}^G C_-(\lambda)_{\xi_\lambda}$ (see (90)). Thus,

$$(\operatorname{gr}^G C_-(\lambda)_{\xi_\lambda}, d_-) = (C_-(\lambda)_{\xi_\lambda}, d_-^{\mathrm{st}}) \tag{98}$$

as complexes. In particular,

$$(E_1^{\bullet,q})_{\xi_\lambda} = H^q(\mathrm{gr}^G C_-(\lambda)^*_{\xi_\lambda}, d) = H^q(C_-(\lambda)^*_{\xi_\lambda}, d^{\mathrm{st}}), \tag{99}$$

because the complex $(C_-(\lambda)_{\xi_\lambda}, d^{\mathrm{st}}_-)$ is a direct sum of finite-dimensional subcomplexes $(C_-(\lambda)^{\mu}_{\xi_\lambda}, d^{\mathrm{st}}_-)$, with $\mu \in \widehat{\mathfrak{h}}^*$.

7.14. Consider the complex

$$G_0 C_-(\lambda)/G_{-1}C_-(\lambda) = C_-(\lambda)/G_{-1}C_-(\lambda).$$

Let $\overline{|\lambda\rangle}$ be the image of $|\lambda\rangle_-$ in $C_-(\lambda)/G_{-1}C_-(\lambda)$. Then, the following hold:

$$\begin{aligned}
&(C_-(\lambda)/G_{-1}C_-(\lambda))_{\xi_\lambda} = \mathbb{C}\overline{|\lambda\rangle},\\
&J^{(v)}_-(n)\overline{|\lambda\rangle} = 0 \quad \text{for } v(n) \in L\mathfrak{g}_{\geq 0} \cap \widehat{\mathfrak{g}}_+,\\
&\psi^{-\alpha}(n)\overline{|\lambda\rangle} = 0 \quad \text{for } \alpha \in \Delta_{>0}, n \geq 0,\\
&\Phi_{-\alpha}(n)\overline{|\lambda\rangle} = 0 \quad \text{for } \alpha \in \Delta_{\frac{1}{2}}, n \geq 1,\\
&J^{(e)}(-1)\overline{|\lambda\rangle} = \psi^{-\theta}(-1)\overline{|\lambda\rangle} = 0,\\
&J^{(h)}_-(0)\overline{|\lambda\rangle} = \langle \lambda, h\rangle \overline{|\lambda\rangle} \quad \text{for } h \in \mathfrak{h}.
\end{aligned} \tag{100}$$

Lemma 7.14.1. *We have*

$$\mathrm{gr}^G C_-(\lambda) = \bigoplus_{\mu \in \widehat{\mathfrak{h}}^*} (C_-(\mu)/G_{-1}C_-(\mu)) \otimes C_-(\lambda)^{\mu}_{\xi_\lambda}$$

as complexes, where $C_-(\lambda)^{\mu}_{\xi_\lambda} = (C_-(\lambda)^{\mu}_{\xi_\lambda}, d^{\mathrm{st}}_-)$.

Proof. Define a linear map $\bigoplus_{\mu \in \widehat{\mathfrak{h}}^*} (C_-(\mu)/G_{-1}C_-(\mu)) \otimes \mathrm{gr}^G C_-(\lambda)^{\mu}_{\xi_\lambda} \to \mathrm{gr}^G C_-(\lambda)$ by

$$\begin{aligned}
&J^{(u_1)}_-(m_1)\dots J^{(u_p)}_-(m_p)\Phi_{-\alpha_1}(n_1)\dots\Phi_{-\alpha_q}(n_q)\psi^{-\beta_1}(s_1)\dots\dots\psi^{-\beta_r}(s_r)\overline{|\mu\rangle}\otimes v\\
&\mapsto J^{(u_1)}_-(m_1)\dots J^{(u_p)}_-(m_p)\Phi_{-\alpha_1}(n_1)\dots\Phi_{-\alpha_q}(n_q)\psi^{-\beta_1}(s_1)\dots\dots\psi^{-\beta_r}(s_r)v,
\end{aligned}$$

where $u_i \in \mathfrak{g}_{\geq 0}$, $\alpha \in \Delta_{\frac{1}{2}}$, $\beta_i \in \Delta_{>0}$, $m_i, n_i, s_i \in \mathbb{Z}$ and $v \in \mathrm{gr}^G C_-(\lambda)^{\mu}_{\xi_\lambda}$. Then, it can be verified this is an isomorphism of complexes (cf. Proposition 6.12 of Ref. [1]). □

7.15. Let

$$(\widehat{\mathfrak{g}}^e)_- \underset{\mathrm{def}}{=} \mathfrak{n}_{0,-}\otimes\mathbb{C}[t^{-1}] \oplus (\mathfrak{h}^e \oplus \mathfrak{n}_{0,+} \oplus \mathfrak{g}_{\frac{1}{2}})\otimes\mathbb{C}[t^{-1}]^{-1} \oplus \mathfrak{g}_1\otimes\mathbb{C}[t^{-1}]t^{-2}.$$

The proof of the following assertion is the same as that of Theorem 4.1 of Ref. [19].

Proposition 7.15.1. *For any* $\lambda \in \widehat{\mathfrak{h}}^*$*, we have* $H^i(C_-(\lambda)/G_{-1}C_-(\lambda)) = \{0\}$ *for* $i \neq 0$ *and the following map defines an isomorphism of* $\mathbb{C}$*-vector spaces:*

$$\begin{array}{ccc}
U((\widehat{\mathfrak{g}}^e)_-) & \to & H^0(C_-(\lambda)/G_{-1}C_-(\lambda)),\\
u_1(n_1)\dots u_r(n_r) & \mapsto & W^{(u_1)}_-(n_1)\dots W^{(u_r)}_-(n_r)\overline{|\lambda\rangle}.
\end{array}$$

Now recall that $G^{-1}C_-(\lambda)^* = D(C_-(\lambda)/G_{-1}C_-(\lambda)) \subset C_-(\lambda)^*$ (see (97)). Because $G^{-1}C_-(\lambda)^*$ is a $C_k(\mathfrak{g})$-submodule of $C_-(\lambda)^*$, it follows that $H^\bullet(G^{-1}C_-(\lambda)^*)$ is a module over $\mathcal{W}_k(\mathfrak{g}, f)$. It is clear that

$$H^\bullet(G^{-1}C_-(\lambda)^*) = \bigoplus_{\xi \leq \xi_\lambda} H^\bullet(G^{-1}C_-(\lambda)^*)_\xi \text{ and } H^\bullet(G^{-1}C_-(\lambda)^*)_{\xi_\lambda} = \mathbb{C}|\lambda\rangle^*. \tag{101}$$

Here with a slight abuse of notation we have denoted the vector in $H^\bullet(G^{-1}C_-(\lambda)^*)_{\xi_\lambda}$ dual to $\overline{|\lambda\rangle}$ by $|\lambda\rangle^*$.

Proposition 7.15.2.

(1) $H^i(G^{-1}C_-(\lambda)^*) = \{0\}$ *for* $i \neq 0$.
(2) *Any nonzero* $\mathcal{W}_k(\mathfrak{g}, f)$*-submodule of* $H^0(G^{-1}C_-(\lambda)^*)$ *contains the canonical vector* $|\lambda\rangle^* \in H^0(G^{-1}C_-(\lambda)^*)_{\xi_\lambda}$.

Proof. (1) It is straightforward to demonstrate using Propositions 7.12.1 and 7.15.1.
(2) For a $\mathcal{W}_k(\mathfrak{g}, f)$-module M, let $S(M)$ be the space of singular vectors:

$$S(M) \underset{\text{def}}{=} \{m \in M \mid \ W^{(u)}(n)m = 0 \text{ for all } v(n) \in (\widehat{\mathfrak{g}}^f)_+\} \subset M.$$

Then, by (101), we have $S(M) \neq \{0\}$ for any nonzero $\mathcal{W}_k(\mathfrak{g}, f)$-submodule M of $H^0(G^{-1}C_-(\lambda)^*)$. Hence, it is sufficient to show that

$$S(H^0(G^{-1}C_-(\lambda)^*)) = \mathbb{C}|\lambda\rangle^*. \tag{102}$$

Note that, by (101), it is obvious that $S(H^0(G^{-1}C_-(\lambda)^*)) \supset \mathbb{C}|\lambda\rangle^*$. Therefore, we have only to show

$$S(H^0(G^{-1}C_-(\lambda)^*)) \subset \mathbb{C}|\lambda\rangle^*. \tag{103}$$

To demonstrate (103), we make use of the filtration $\{F_p\mathcal{W}_k(\mathfrak{g}, f)\}$ of $\mathcal{W}_k(\mathfrak{g}, f)$ given in Theorem 5.5.1, described by (63). First, set $F_{-1}H^0(C_-(\lambda)/G_{-1}C_-(\lambda)) = \{0\}$ and

$$\begin{aligned} &F_pH^0(C_-(\lambda)/G_{-1}C_-(\lambda)) \\ &\underset{\text{def}}{=} \operatorname{span}\{W_-^{(u_1)}(n_1)\dots W_-^{(u_r)}(n_r)\overline{|\lambda\rangle} \mid u_i \in \mathfrak{g}^e_{s_i},\ \sum s_i \leq p\} \quad \text{for } p \geq 0. \end{aligned}$$

Next, define

$$\begin{aligned} &F_pH^0(G^{-1}C_-(\lambda)^*) \\ &= D\left(H^0(C_-(\lambda)/G_{-1}C_-(\lambda))/F_{-p}H^0(C_-(\lambda)/G_{-1}C_-(\lambda))\right) \quad \text{for } p \leq 1. \end{aligned}$$

(Recall that $H^0(G^{-1}C_-(\lambda)^*) = D(H^0(C_-(\lambda)/G_{-1}C_-(\lambda))))$. Then, we have

$$\begin{aligned} &\dots \subset F_pH^0(G^{-1}C_-(\lambda)^*) \subset F_{p+1}H^0(G^{-1}C_-(\lambda)^*) \subset \dots \\ &\quad \dots \subset F_0H^0(G^{-1}C_-(\lambda)^*) \subset F_1H^0(G^{-1}C_-(\lambda)^*) = H^0(G^{-1}C_-(\lambda)^*), \end{aligned} \tag{104}$$

$$\bigcap F_pH^0(G^{-1}C_-(\lambda)^*) = \{0\}, \tag{105}$$

$$F_p\mathcal{W}_k(\mathfrak{g}, f) \cdot F_qH^0(G^{-1}C_-(\lambda)^*) \subset F_{p+q}H^0(G^{-1}C_-(\lambda)^*), \tag{106}$$

where $F_p\mathcal{W}_k(\mathfrak{g}, f) \cdot F_qH^0(G^{-1}C_-(\lambda)^*)$ denotes the span of the vectors $Y_n(v)w$ with $v \in F_p\mathcal{W}_k(\mathfrak{g}, f),\ w \in F_qH^0(G^{-1}C_-(\lambda)^*)$ and $n \in \mathbb{Z}$.

By (106), the corresponding graded space $\operatorname{gr}_F H^0(G^{-1}C_-(\lambda)^*)$, is a module over $\operatorname{gr}_F \mathcal{W}_k(\mathfrak{g}, f) = V_k^\natural(\mathfrak{g}^f)$. In particular, $\operatorname{gr}_F H^0(G^{-1}C_-(\lambda)^*)$ is a module over the Lie

algebra $(\widehat{\mathfrak{g}}^f)_+$. (Observe $(\widehat{\mathfrak{g}}^f)_+$ is generated by the image of $W^{(u)}(n)$, $u(n) \in (\widehat{\mathfrak{g}}^f)_+$.) Moreover, from (the proof of) Proposition 7.15.1, it follows that

$$\mathrm{gr}_F\, H^0(G^{-1}C_-(\lambda)^*) \cong D(U(\widehat{\mathfrak{g}}^e)_-) \quad \text{as } (\widehat{\mathfrak{g}}^f)_+\text{-modules}, \tag{107}$$

where $(\widehat{\mathfrak{g}}^f)_+$ acts on $D(U(\widehat{\mathfrak{g}}^e)_-)$ by $(u(n)g)(v) = g(u^t(-n)v)$, where $u \in \mathfrak{g}^f$, $v \in U(\widehat{\mathfrak{g}}^e)$, $g \in D(U(\widehat{\mathfrak{g}}^e))$. Hence, it follows that

$$H^0((\widehat{\mathfrak{g}}^f)_+, \mathrm{gr}_F\, H^0(G^{-1}C_-(\lambda)^*)) = \mathbb{C}(\text{the image of } |\lambda\rangle^*).$$

But this implies that $S(H^0(G^{-1}C_-(\lambda)^*))) \subset \mathbb{C}|\lambda\rangle^*$. This completes the proof. □

7.16.

Theorem 7.16.1. *Suppose that $\langle \lambda, \alpha_0^\vee \rangle \notin \{0, 1, 2, \dots\}$. Then, $H^i(M(\lambda)^*) = \{0\}$ for $i \neq 0$ and $H^0(M(\lambda)^*) = H^0(G^{-1}C_-(\lambda)^*)$ as $\mathcal{W}_k(\mathfrak{g}, f)$-modules. In particular, any nonzero $\mathcal{W}_k(\mathfrak{g}, f)$-submodule of $H^0(M(\lambda)^*)$ contains the canonical vector $|\lambda\rangle^*$.*

Proof. We have $\bar{M}_{\mathfrak{sl}_2}(\langle \lambda, \alpha_0^\vee \rangle)^* = \bar{M}_{\mathfrak{sl}_2}(\langle \lambda, \alpha_0^\vee \rangle)$ by assumption. Also, we have $H^i(C_-(\lambda)_{\xi_\lambda}, d_-^{\mathrm{st}}) \cong H^i(\mathbb{C}e, \bar{M}_{\mathfrak{sl}_2}(\langle \lambda, \alpha_0^\vee \rangle))$. This can be demonstrated in the same manner as Lemma 4.6.1. Hence, it follows that

$$H^i(C_-(\lambda)^\mu_{\xi_\lambda}, d_-^{\mathrm{st}}) = \begin{cases} \mathbb{C} & (i = 0 \text{ and } \mu = \lambda), \\ \{0\} & (\text{otherwise}). \end{cases}$$

Therefore, by Lemma 7.14.1 and Proposition 7.15.1, we have

$$H^i(\mathrm{gr}^G C_-(\lambda)) = \begin{cases} H^0(C_-(\lambda)/G_{-1}C_-(\lambda)) & (i = 0), \\ \{0\} & (i \neq 0). \end{cases}$$

Hence, by Proposition 7.12.1, we have

$$H^i(\mathrm{gr}^G C_-(\lambda)^*) = \begin{cases} H^0(G^{-1}C_-(\lambda)^*) & (i = 0) \\ \{0\} & (i \neq 0). \end{cases}$$

This implies that our spectral sequence collapses at $E_1 = E_\infty$, and therefore

$$H^i(M(\lambda)^*) = \begin{cases} H^0(G^{-1}C_-(\lambda)^*) & (i = 0) \\ \{0\} & (i \neq 0) \end{cases}$$

as vector spaces. But the isomorphism $H^0(G^{-1}C_-(\lambda)^*) \cong H^0(M(\lambda)^*)$ is induced by the $C_k(\mathfrak{g})$-module homomorphism $G^{-1}C_-(\lambda)^* \hookrightarrow C_-(\lambda)^*$, and hence it is a $\mathcal{W}_k(\mathfrak{g}, f)$-homomorphism. □

7.17. We finally consider the case in which λ is a general element of $\widehat{\mathfrak{h}}_k^*$. By Proposition 7.15.1, we have

$$H^0(G^{-1}C_-(\mu)^*) \cong H^0(G^{-1}C_-(\mu')^*) \tag{108}$$

as $\mathbb{C}$-vector spaces for any $\mu, \mu' \in \widehat{\mathfrak{h}}^*$. With (108), it follows from (95), Proposition 7.12.1 and Lemma 7.14.1 that we have the isomorphism

$$\begin{aligned} E_1^{\bullet,q} &\cong H^0(G^{-1}C_-(\lambda)^*) \otimes H^q(C_-(\lambda)^*_{\xi_\lambda}, d^{\mathrm{st}}) \\ &= H^0(G^{-1}C_-(\lambda)^*) \otimes (E_1^{\bullet,q})_{\xi_\lambda} \end{aligned} \tag{109}$$

of complexes, where the differential d_1 acts on the first factor $H^0(G^{-1}C_-(\lambda)^*)$ trivially. This induces the isomorphisms

$$(E_r, d_r) \cong (H^0(G^{-1}C_-(\lambda)^*)\otimes(E_r)_{\xi_\lambda}, 1\otimes d_r) \tag{110}$$

inductively for all $r \geq 1$. Therefore, we obtain

$$E_\infty \cong H^0(G^{-1}C_-(\lambda)^*)\otimes(E_\infty)_{\xi_\lambda}. \tag{111}$$

On the other hand, by (96), we have

$$(E_\infty)_{\xi_\lambda} = H^\bullet(M(\lambda)^*)_{\xi_\lambda}.$$

Hence, by (111) and Proposition 4.8.1 (3), we have proved Theorem 6.6.1, as desired. ■

Appendix A. The setting for $A(1,1)$

In this appendix we summarize the change of the setting for the $A(1,1)$ case.

A.1. **The setting for $\mathfrak{g}$.** Let $\mathfrak{g} = \mathfrak{sl}(2|2)/\mathbb{C}I$ and let $\mathfrak{h}'$ be the Cartan subalgebra of $\mathfrak{g}$ containing x. Let $\mathfrak{a} \underset{\text{def}}{=} \mathfrak{gl}(2|2)/\mathbb{C}I$. Then, $\mathfrak{g} = [\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{a}$ containing $\mathfrak{h}'$. (So $\dim \mathfrak{h} = 3$.) Then, $[\mathfrak{h}, \mathfrak{g}] = \mathfrak{g}$ and we have $\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{h}^*} \mathfrak{g}^\alpha$, where $\mathfrak{g}^\alpha = \{u \in \mathfrak{g} \mid [h, u] = \langle\alpha, h\rangle u \text{ for all } h \in \mathfrak{h}\}$. Define the set Δ as the subset of $\mathfrak{h}^*$ consisting of elements α such that $\mathfrak{g}^\alpha \neq \{0\}$. Then, $\dim \mathfrak{g}^\alpha = 1$ for all $\alpha \in \Delta$. The remaining setting is the same as in the other cases.

A.2. **The setting for $\widehat{\mathfrak{g}}$.** Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra associated with $\mathfrak{g} = \mathfrak{sl}(2|2)/\mathbb{C}I$ defined by (7). Let $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}}' \oplus \widehat{\mathfrak{n}}_+$ be the standard triangular decomposition, where $\widehat{\mathfrak{h}}' = \mathfrak{h}' \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D}$ is the standard Cartan subalgebra of $\widehat{\mathfrak{g}}$. Next, define the commutative Lie algebra $\widehat{\mathfrak{h}}$ by $\widehat{\mathfrak{h}} \underset{\text{def}}{=} \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}\mathbf{D}$, where $\mathfrak{h}$ is as above. Then, the action of $\mathfrak{h}$ on $\mathfrak{g}$ naturally extends to the action of $\widehat{\mathfrak{h}}$ on $\widehat{\mathfrak{g}}$. This gives the space

$$\widetilde{\mathfrak{g}} \underset{\text{def}}{=} \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+ \tag{112}$$

a Lie superalgebra structure such that $\widehat{\mathfrak{g}} = [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$, and $\widehat{\mathfrak{h}}$ is a Cartan subalgebra of $\widetilde{\mathfrak{g}}$. Now define $\widehat{\Delta} \subset \widehat{\mathfrak{h}}^*$ as the set of roots of $\widetilde{\mathfrak{g}}$ and $\widehat{\Delta}_+ \subset \widehat{\Delta}$ as the set of positive roots of $\widetilde{\mathfrak{g}}$ (according to the triangular decomposition (112)). Also, we define $\widehat{Q} = \sum_{\alpha\in\Delta} \mathbb{Z}\alpha \subset \widehat{\mathfrak{h}}^*$, $\widehat{Q}_+ = \sum_{\alpha\in\widehat{\Delta}_+} \mathbb{Z}_\alpha \subset \widehat{Q}$ and a partial ordering $\mu \leq \lambda$ on $\widehat{\mathfrak{h}}^*$ by $\lambda - \mu \in \widehat{Q}_+$.

Next, we replace $\widehat{\mathfrak{g}}$-modules by $\widetilde{\mathfrak{g}}$-modules. In particular, we define the category $\mathcal{O}_k$ as the full subcategory of the category of $\widetilde{\mathfrak{g}}$-modules satisfying the conditions of Subsection 2.12. Note that the simple $\widetilde{\mathfrak{g}}$-module $L(\lambda)$, with $\lambda \in \widehat{\mathfrak{h}}^*$, is already irreducible as a $\widehat{\mathfrak{g}}$-module. This fact can be seen using the argument of the proof of Theorem 6.7.4. The remaining setting is the same as in the other cases.

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